Symmetries, Outer Space, & the Outer Automorphism Group of the Free Group

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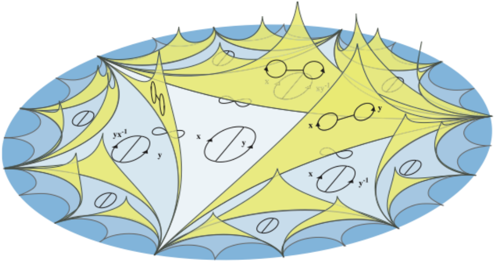


I. Groups

Groups

Familiar: **Integers** $\mathbb{Z}$ with **addition** $+$:

1. Have an identity: $(-5) + 0 = -5, \quad 0 + 1002 = 1002$
2. Have inverses: $7 + (-7) = 0, \quad (-22) + 22 = 0$
3. Are associative: $(5 + (-3)) + 12 = 5 + ((-3) + 12)$

So $(\mathbb{Z}, +)$ example of:

Definition (Group)

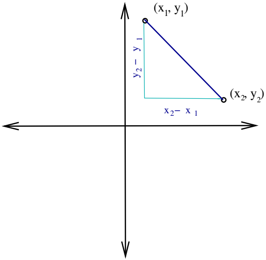
Collection of objects G with **Binary operation** $\star$ where:

1. G has an identity e: $a \star e = a$, $e \star a = a$ for all $a \in G$.
2. Each $a \in G$ has an inverse a^{-1}: $a \star a^{-1} = e = a^{-1} \star a$.
3. Associativity: All $a, b, c \in G$ satisfy: $(a \star b) \star c = a \star (b \star c)$.

II. Symmetries, Isometry Groups, & the Spaces they Act On

Elements of groups can be functions!

Integers are actually distance-preserving functions, i.e. **isometries**!



$$d_{\mathbb{R}^2}(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

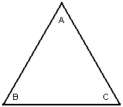
- $a \in \mathbb{Z} \rightsquigarrow$ isometry of $\mathbb{R}^2$: $(x, y) \mapsto (x + a, y)$

  $(x, y) \mapsto (x + 3, y)$ then $(x, y) \mapsto (x + 2, y)$, same as $(x, y) \mapsto (x + (3 + 2), y)$!

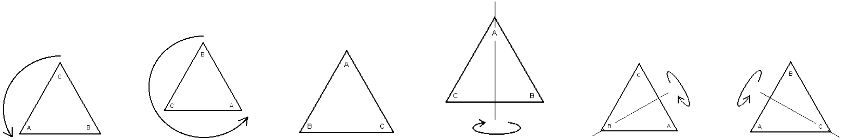
- $(a, b) \in \mathbb{Z} \times \mathbb{Z} \rightsquigarrow$ isometry of $\mathbb{R}^2$: $(x, y) \mapsto (x + a, y + b)$

- **Isometry group**: All rotations, reflections, translations, glide reflections

Symmetries of a Triangle



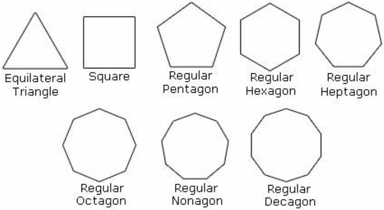
Symmetry (isometry) group?



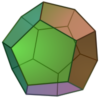
Every symmetry is a composition of the same rotation & flip!

$\frac{i}{8}\pi$

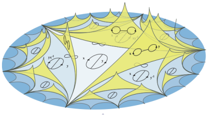
Can do with all kinds of polygons...



Or more general polytopes...



Or even...



III. Folding things up (Quotients)

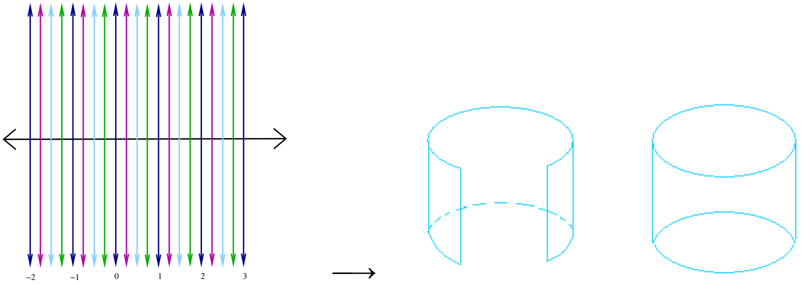
Quotients: The cylinder

Group of isometries $\rightsquigarrow$ Quotient space

(Identifying each point with all its images)

Group: Isometries of $\mathbb{R}^2$ defined by $(x, y) \mapsto (x + a, y)$ with $a \in \mathbb{Z}$

Quotient:

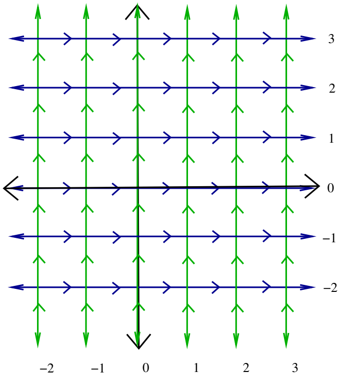


Quotients: The torus

Group: Isometries of $\mathbb{R}^2$ defined by $(x, y) \mapsto (x + a, y + b)$ for $a, b \in \mathbb{Z}$

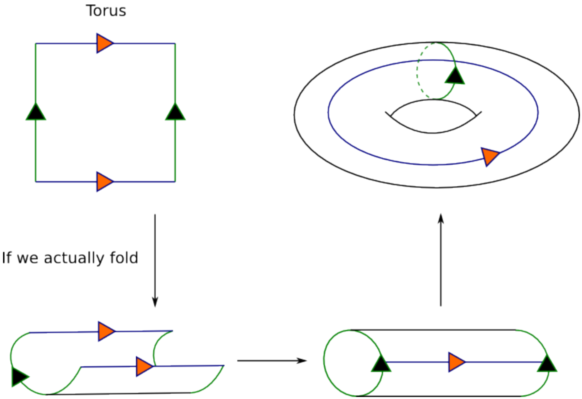
Quotient:

This group takes each square in picture to each other square in picture



Quotients: The torus

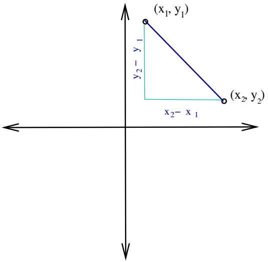
How taking the quotient leads to a torus:



IV. The ways we travel in $\mathbb{R}^2$

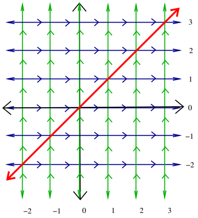
Geodesics in $\mathbb{R}^2$

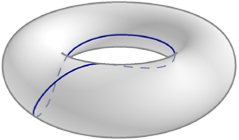
The shortest path between points is always a straight line ($y = mx + b$):

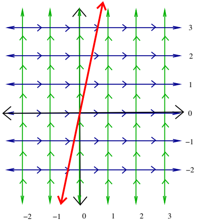


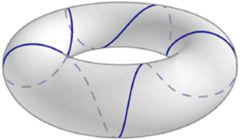
$$d_{\mathbb{R}^2}(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Geodesics in $\mathbb{R}^2 \rightsquigarrow$? in Torus



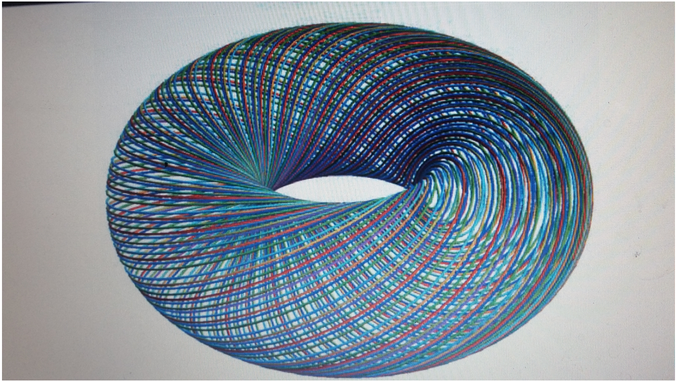






Geodesics in $\mathbb{R}^2 \rightsquigarrow$? in Torus

What if slope irrational (not $\frac{a}{b}$ with $a, b \in \mathbb{Z}$)?



This geodesic image is **dense**!
(Passes infinitely close to each point)

V. Special Group: Out(F_r)

Recall Groups

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The Free Group F_r

$F_r = \langle x_1, \ldots, x_r \rangle$ **rank-r free group**

Example: $F_3 = \langle x_1, x_2, x_3 \rangle$

Elements of the group look like:

$x_2 x_1 x_3^{-1}$ or $x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_1$ or $x_3^{-1} x_2 x_1$

What elements of the group don't look like:

$x_1 x_3^{-1} x_3 x_2 x_1 x_3^{-1}$ (write instead $x_1 x_2 x_1 x_3^{-1}$)

How multiply elements of group:

$x_1 x_3^{-1} * x_3 x_2 x_1 x_3^{-1} = x_1 x_3^{-1} x_3 x_2 x_1 x_3^{-1} = x_1 x_2 x_1 x_3^{-1}$

Automorphisms of F_r

Interested in permutations φ of F_r *preserving the structure*, i.e. want

$$\varphi(a \star b) = \varphi(a) \star \varphi(b)$$

They look like "replacement" functions, e.g.

- $x_2 \rightsquigarrow x_2 x_3^{-1}$
- (& $x_2^{-1} \rightsquigarrow x_3 x_2^{-1}$).

So $\varphi(x_2 x_1 x_3^{-1}) = x_2 x_3^{-1} x_1 x_3^{-1}$

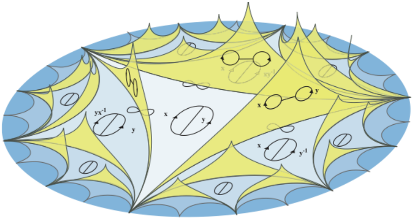
Or compositions of these kinds of functions.

Out(F_r) as an isometry group

Out(F_r) is the isometry group of Culler-Vogtman Outer Space...

Will be a simplicial complex where each point represents a graph!

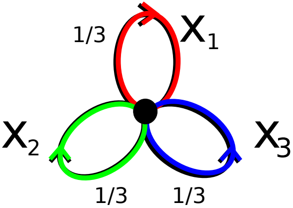
VI. "Inspiration from Above": Outer Space

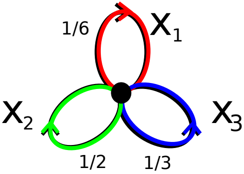


Outer space CV_r

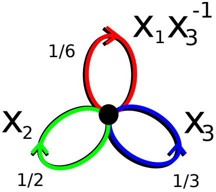
Points in CV_r are *marked*, *metric*, *graphs*:

Most basic point:

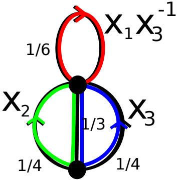




Change lengths on edges



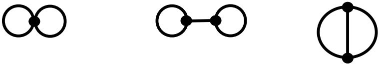
Apply automorphism



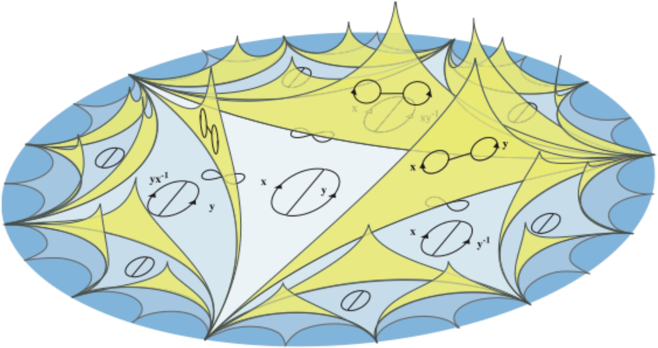
"Blow up" vertex

Outer Space in Rank 2 (CV_2)

The graphs Γ with 2 loops (i.e. $\pi_1(\Gamma) = F_2$):

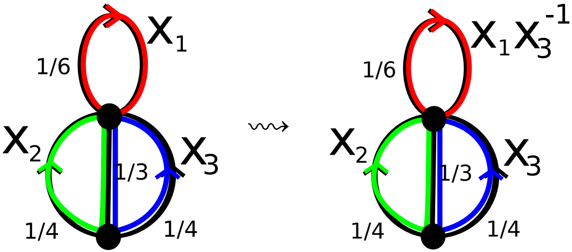


CV_2:



The symmetries (isometries) of CV_r?

Just apply an automorphism!



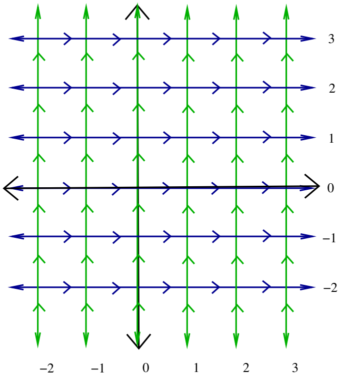
VII. Folding Up Outer Space

Recall torus

Group: Isometries of $\mathbb{R}^2$ defined by $(x, y) \mapsto (x + a, y + b)$ for $a, b \in \mathbb{Z}$

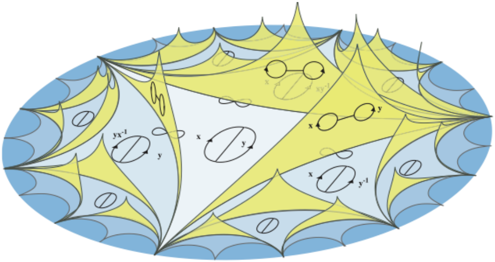
Quotient:

This group takes each square in picture to each other square in picture



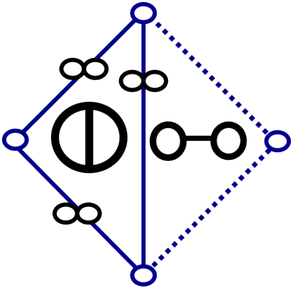
Quotient of Outer Space: $CV_2 / \mathrm{Out}(F_2)$

Recall CV_2:



Quotient of Outer Space: $CV_2/\operatorname{Out}(F_2)$

$CV_2/\operatorname{Out}(F_2)$ looks like



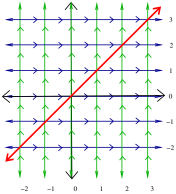
but with edges identified:

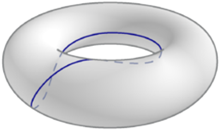
VIII. Traveling in Outer Space

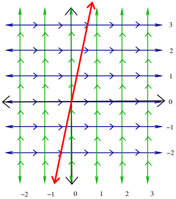
Geodesics in Quotient

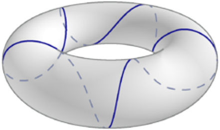
Question: What do geodesics in quotient of Outer Space look like?

Recall Torus





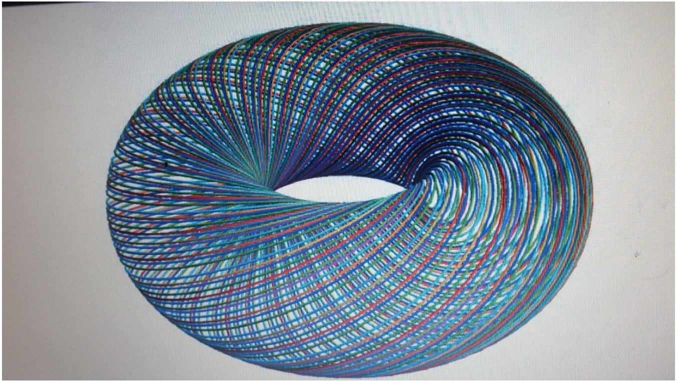




Also have geodesics closing up like this in quotient of Outer Space.

Recall Torus

For slope irrational (not $\frac{a}{b}$ with $a, b \in \mathbb{Z}$):



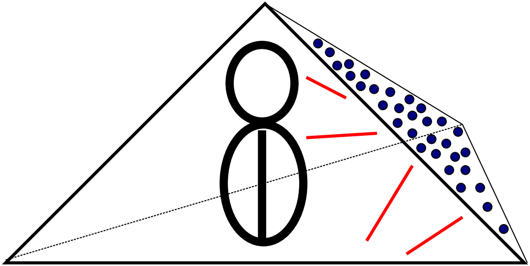
This geodesic image is **dense**! (Passes infinitely close to each point)

Does Outer Space have geodesics like this?

Dense Geodesics

Theorem (Theorem; Algom-Kfir, Pfaff)

Yes!!! (The quotient of Outer Space has dense geodesics.)



Thank you!