A Nielsen-Thurston inspired story of iterating free group automorphisms and efficiently deforming graphs

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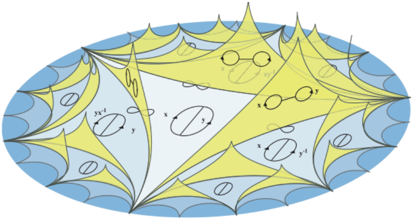
Colloquium, February 2018



Is A Tale With Two Surprisingly Interconnected Themes

What happens when you iterate a free group automorphism?

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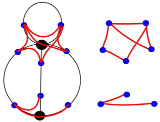
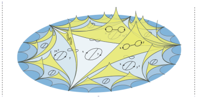
What happens when you efficiently deform a metric graph?

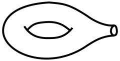
- Outer automorphism invariants



- Geodesics in Culler-Vogtmann Outer Space



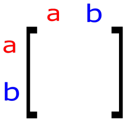
The Backstory (will return to)

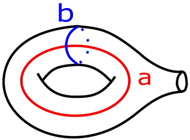
$GL_2(Z)$ ≅ MCG() ≅ $Out(F_2)$

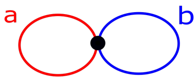
2x2 integer matrices of determinant +/- 1

Homeo() / Homotopy

$Aut(F_2) / Inn(F_2)$







The Free Group F_r

$F_r = \langle x_1, \ldots, x_r \rangle$ **rank-r free group**

Example: $F_3 = \langle x_1, x_2, x_3 \rangle$

Some elements:

$x_2 x_1 x_3^{-1}$ or $x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_1$ or $x_3^{-1} x_2 x_1$

Multiplication:

$x_1 x_3^{-1} * x_3 x_2 x_1 x_3^{-1} = x_1 x_3^{-1} x_3 x_2 x_1 x_3^{-1} = x_1 x_2 x_1 x_3^{-1}$

Outer Automorphism Group of the Free Group $Out(F_r)$

$F_r = \langle x_1, \ldots, x_r \rangle$ rank-r free group

Definition

$$Out(F_r) = \frac{Aut(F_r)}{Inn(F_r) = \{\varphi_a \mid \varphi_a(b) = aba^{-1} \ \forall \ a, b \in F_r\}}$$

To define $\Phi \in Aut(F_r)$, just need to describe images of generators:

$$\Phi = \begin{cases} x_1 \mapsto x_1 x_3^{-1} \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_2 \end{cases}$$

Nielsen proved they're compositions of these "replacement" functions, e.g.

- $x_2 \rightsquigarrow x_2 x_3^{-1}$ (& $x_2^{-1} \rightsquigarrow x_3 x_2^{-1}$).

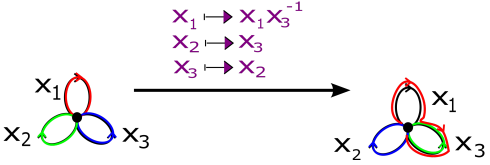
$\varphi(x_2 x_1) = x_2 x_3^{-1} x_1$ & $\varphi(x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_2) = x_3 x_1^{-1} x_3 x_2^{-1} x_3^{-1} x_2 x_3^{-1}$

Outer Automorphism Group of the Free Group $Out(F_r)$

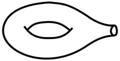
To utilize work of Nielsen, Skora, Stallings, Whitehead, and Bestvina-Feighn-Handel we view

$$\Phi = \begin{cases} x_1 \mapsto x_1 x_3^{-1} \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_2 \end{cases}$$

as a homotopy equivalence of graphs:



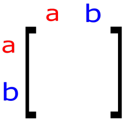
The Backstory

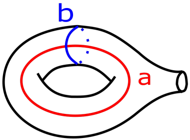
$GL_2(Z) \cong MCG($  $) \cong Out(F_2)$

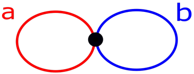
2x2 integer matrices
of determinant +/- 1

Homeo() / Homotopy

$Aut(F_2) / Inn(F_2)$

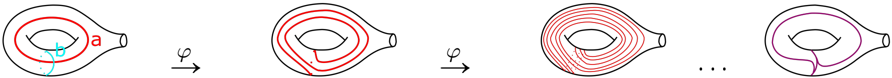






Nielsen-Thurston studied asymptotic dynamical invariants

For φ a generic surface homeo, repeated application of φ to any curve limits on the same object...



$$\begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}} \begin{bmatrix}2\\1\end{bmatrix} \xrightarrow{\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}} \begin{bmatrix}5\\3\end{bmatrix} \quad \cdots \quad k\begin{bmatrix}\frac{1+\sqrt{5}}{2}\\1\end{bmatrix}$$

Some important "conjugacy class" invariants:

$GL(2,\mathbb{Z})$	$MCG(\Sigma_{1,1})$	$Out(F_2)$
• Eigenvector	• Lamination	• Lamination
		• Axis bundle
	• Indices / IWG	• Indices / IWG
• Dominant eigenval	• Stretch factor	• Stretch factor

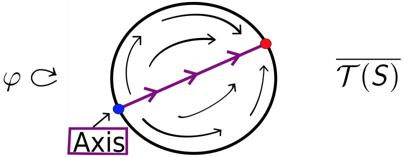
$\mathcal{T}(S)$: Deformation space of hyp. metrics on a surface S

• Amazingly, the space of metrics is itself a metric space &...

• [Royden] For closed surface S: $Isom(\mathcal{T}(S)) \cong MCG(S)$

• [Thurston] $\mathcal{T}(S)$ is compactified by projective measured laminations on S & $\overline{\mathcal{T}(S)}$ is a ball

• Bers, Marden, Masur, Strebel, Thurston, & others connected mapping class group invariants with geodesics in $\mathcal{T}(S)$

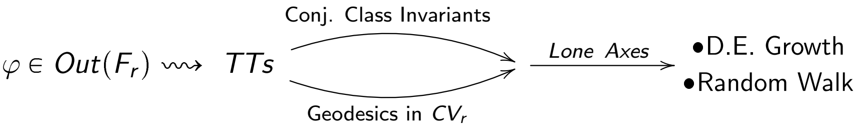


The $Out(F_r)$ Tale: Interconnected Goals & Strategy

Interconnected Goals:

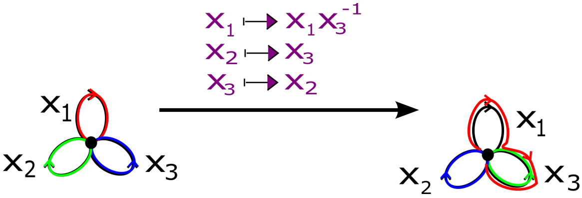
1. Understanding $Out(F_r)$ conjugacy class invariants
2. Understanding Geodesics in Outer Space CV_r

Strategy/Outline:



Train Track Representatives (Bestvina-Handel)

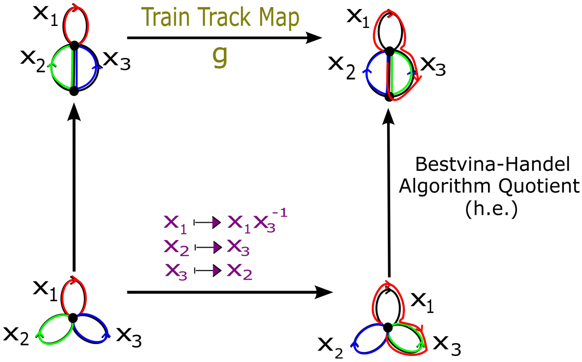
Recall: $\varphi \in Out(F_r)$ always have topological representatives:



- But iteration may lead to cancellation on edge interiors

Train Track Representatives (Bestvina-Handel)

Nice $\varphi \in Out(F_r)$ have **train track representatives** $g\colon \Gamma \xrightarrow{h.e.} \Gamma$



No cancellation on edge interiors even after iteration!

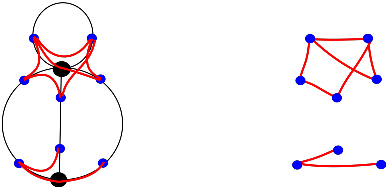
$Out(F_r)$ Conjugacy Class Invariants: $\mathcal{IW}(\varphi)$, $\lambda(\varphi)$

Idea in analogy with Nielsen-Thurston setting:

- Iterate loops (*in graph*) $\rightsquigarrow$ Lamination leaves

$a \overset{\varphi}{\mapsto} abbab \overset{\varphi}{\mapsto} abbabbababbabbababbababbabbababbab$

- $\mathcal{IW}(\varphi)$: Record at vertices how lamination leaves enter & leave



- $\lambda(\varphi)$: Record how lamination leaves stretched by φ (equivalently how curves asymptotically stretched)

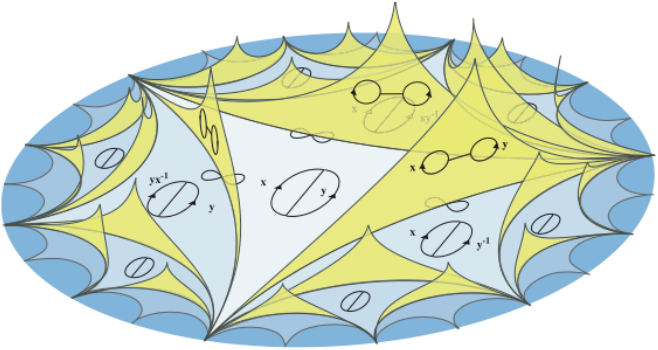
Strategy/Outline:

$\varphi \in Out(F_r) \rightsquigarrow$ TTs $\xrightarrow{\text{Conj. Class Invariants}}$ / $\xrightarrow{\text{Geodesics in } CV_r}$ $\xrightarrow{\textit{Lone Axes}}$

- D.E. Growth
- Random Walk

$Out(F_r)$ & the Deformation Space of Metric Graphs

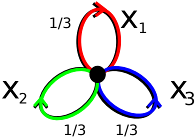
$Out(F_r)$ is the isometry group for a deformation space of metric graphs, Culler-Vogtmann Outer Space CV_r



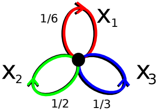
Outer Space CV_r

Points in CV_r are *marked*, *metric*, *graphs*:

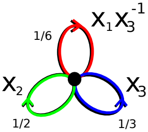
Most basic point:



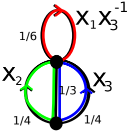
Can additionally:



Change lengths on edges



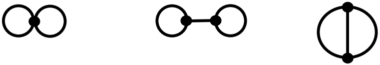
Apply automorphism



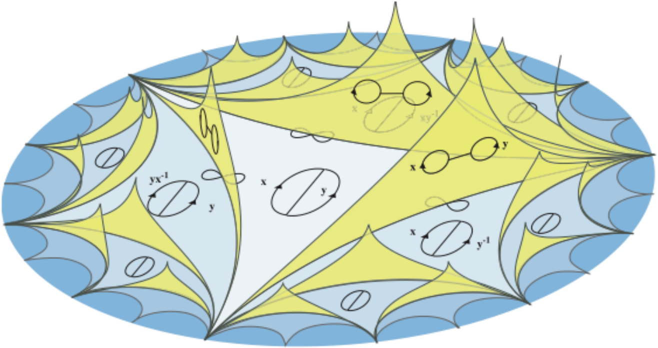
“Blow up” vertex

Outer Space in Rank 2 (CV_2)

The graphs Γ with $\pi_1(\Gamma) = F_2$:

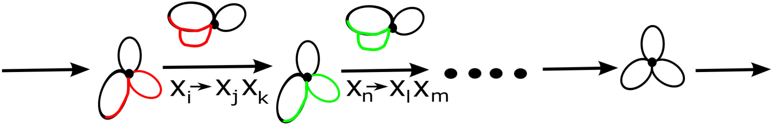


CV_2:



TT Maps $g \rightsquigarrow$ Geodesics in CV_r

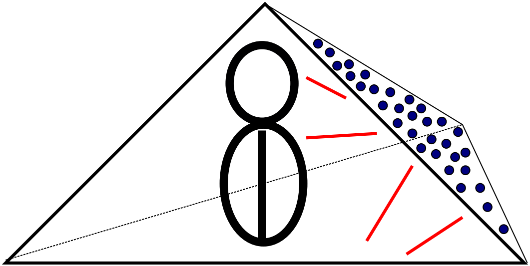
- Stallings decomposes g as a sequence of "folds"
- Skora made process continuous $\rightsquigarrow$ geodesic in CV_r



Complicated Geodesics Exist: Dense Geodesics

Theorem (Main Theorem I; Algom-Kfir, Pfaff)

For each $r \geqslant 2$,
$\exists$ a geodesic fold ray in CV_r w/ dense image under
$CV_r \xrightarrow{quotient} CV_r/Out(F_r)$.



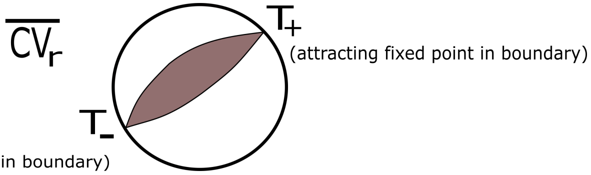
Strategy/Outline:

$\varphi \in Out(F_r) \rightsquigarrow TTs$ $\xrightarrow[\text{Geodesics in } CV_r]{\text{Conj. Class Invariants}}$ $\xrightarrow{\textit{Lone Axes}}$ •D.E. Growth •Random Walk

Surprising Connection: Lone Axes in Outer Space

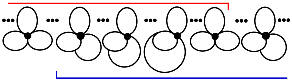
Definition (Axis bundle $\mathcal{A}_\varphi$ for nice $\varphi \in Out(F_r)$ (Handel, Mosher))

$\mathcal{A}_\varphi = \overline{\{\text{Fold line geodesics for tt reps of } \varphi^k \textit{ with } k > 0\}} \subset CV_r$



Theorem (Main Theorem II; Mosher, Pfaff)

(Algorithmically checkable) $\mathcal{IW}(\varphi)$ condition for when $\mathcal{A}_\varphi$ a "lone axis"

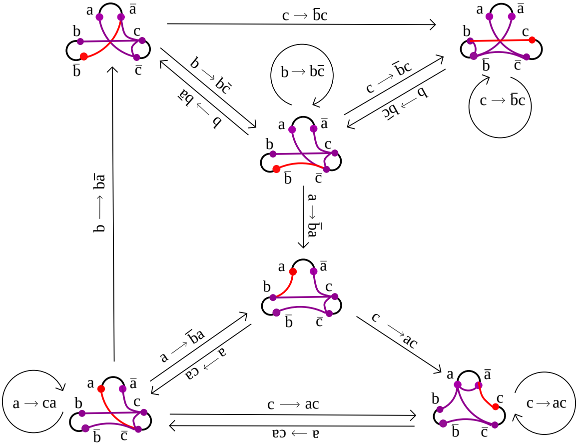


- Exist (Pfaff), many (Kapovich-Pfaff), different kinds (Coulbois-Lustig)

Lone Axes are Beautiful

When additionally $\mathcal{IW}(\varphi)$ connected:

"Lone Axes" give infinite paths in the tt automata of my thesis



Strategy/Outline:

$\varphi \in Out(F_r) \rightsquigarrow$ TTs — Conj. Class Invariants / Geodesics in CV_r → *Lone Axes* →
- D.E. Growth
- Random Walk

Double Exponential Growth

•[Margulis, '69,'70] In hyp. manifold # of closed curves of bdd lngth grows exponentially w. the lngth bound.

•[Eskin, Mirzakhani, '11] In $\mathcal{T}(S)/MCG(S)$, # of closed curves of bdd lngth grows exponentially w. the lngth bound.

• $\mathfrak{N}_r(R) := \#\{[\varphi] \mid \varphi \text{ f.i., } log(\lambda(\varphi)) < R\}$

Theorem (Main Theorem IV; I. Kapovich, Pfaff)

For $r \geqslant 3$, $\exists$ constants $a > 1, b > 1, c > 1, M$ s.t. $\forall\ R > M$:

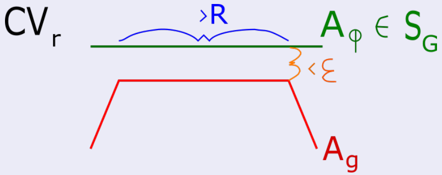
$$c^{e^R} \leqslant \mathfrak{N}_r(R) \leqslant a^{b^R}$$

• Corollary: # of closed curves grows double exponentially!!!

Test for generic behavior (Stable Strata)

Theorem (Main Theorem III; Algom-Kfir, I. Kapovich, Pfaff)

- *Stratify geodesics by* $\mathcal{IW}(\varphi)$
- $\exists$ *lone axis "stable strata"* S_G*:*



$$\Longrightarrow \quad A_g \in S_G$$

- The lies make this interesting!

Unhatched Egg: Random Walks

If you would like this slide, please contact Catherine Pfaff directly.

Extended Program: Dynamical properties of space of geodesics

- Dense geodesic ray, stable strata, and growth rate work all give hints of dynamical properties of spc of geodesics in CV_r
- What kind of dynamical theory can be built on spc of geodesics in CV_r & what does this tell us about $Out(F_r)$?

Broader Program: Deformation Spaces

- How much of Nielsen-Thurston story holds for the space of convex projective structures on a surface?
  (Work w. Cooper on associating foliations to hyp metrics is bridge)

- Billera, Holmes, & Vogtmann define & study geometry of spc of phylogenetic trees.
  Can constructing other deformation spaces help to understand other biological or technological settings?

Thank you!