Lecture Notes : $\mathbb{R}$-trees (Thiery Coulbois) 24/06/2013

Trees (with actions by isometries)

1) Outer Spaces | examples of trees
   $\mathbb{R}$-trees
   ∂CV_N

2) Rip's theorem

3) laminations / hearts / botany of trees

Γ finite graph, $\pi_1(\Gamma) = F_N$ $\rightsquigarrow$ the universal cover of Γ is a simplicial tree.

$F_N \circlearrowright \tilde{\Gamma}$ freely (simplicial)
$\uparrow$
by Deck transformations.

Metric spaces:

$l : E(\Gamma) \longrightarrow \mathbb{R}_{>0} \rightsquigarrow l : E(\tilde{\Gamma}) \longrightarrow \mathbb{R}_{>0}$

Γ: metric graph

$\tilde{\Gamma}$: metric space, geodesic

triangles are tripods

realize each edge as $[0, l(e)]$

Graph [Serre] $E(\Gamma)$ edges, $V(\Gamma)$ verticles

$t, i : E(\Gamma) \longrightarrow V(\Gamma)$ (terminal points, initial points)

$\bar{\ } : E(\Gamma) \longrightarrow E(\Gamma)$ involution without fix points.

$i(\bar{e}) = t(e)$

In $\tilde{\Gamma}$ there exists a unique geodesic path $[x, y]$ between any two points.

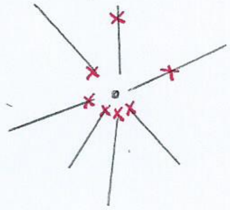
<u>Gromov product</u>: $C_x(y,z) = -\frac{1}{2}(d(y,z) - d(x,y) - d(x,z))$

$\mathbb{R}$-trees: metric space, geodesic

0-hyperbolic: $\forall x, y, z, t$

$$C_x(y,z) \geq \min(C_x(y,t), C_x(z,t))$$

② simplicial topology $\neq$ metric topology



vertex with ∞ valence

not conjugacy to the simplicial topology

Isometries of trees:

$f: T \to T$ isometry.

1) f is elliptic $\quad Fix(f) \neq \emptyset \iff \|f\|_T = 0$

$\Rightarrow Fix(f)$ is closed subtree of T

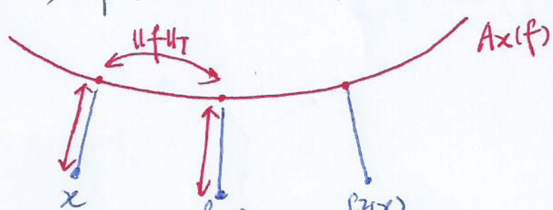
or

2) f is hyperbolic $\iff \|f\|_T > 0$

$\|f\|_T = \inf_{x \in T} d(x, f(x))$ translation length "minimal displacement"

$Ax(f) = \{x \in T \mid d(x, f(x)) = \|f\|_T\}$

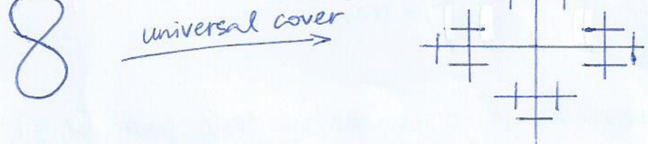
hyperbolic $\Rightarrow$ f acts on $Ax(f)$ as a translation (by $\|f\|_T$)



$d(x, f(x)) = 2d(x, Ax(f)) + \|f\|_T$

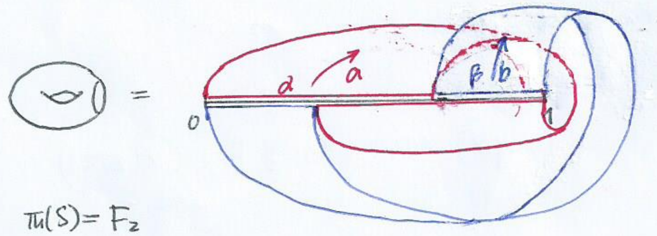
Examples:

Simplicial trees:



Interval exchange transformation:

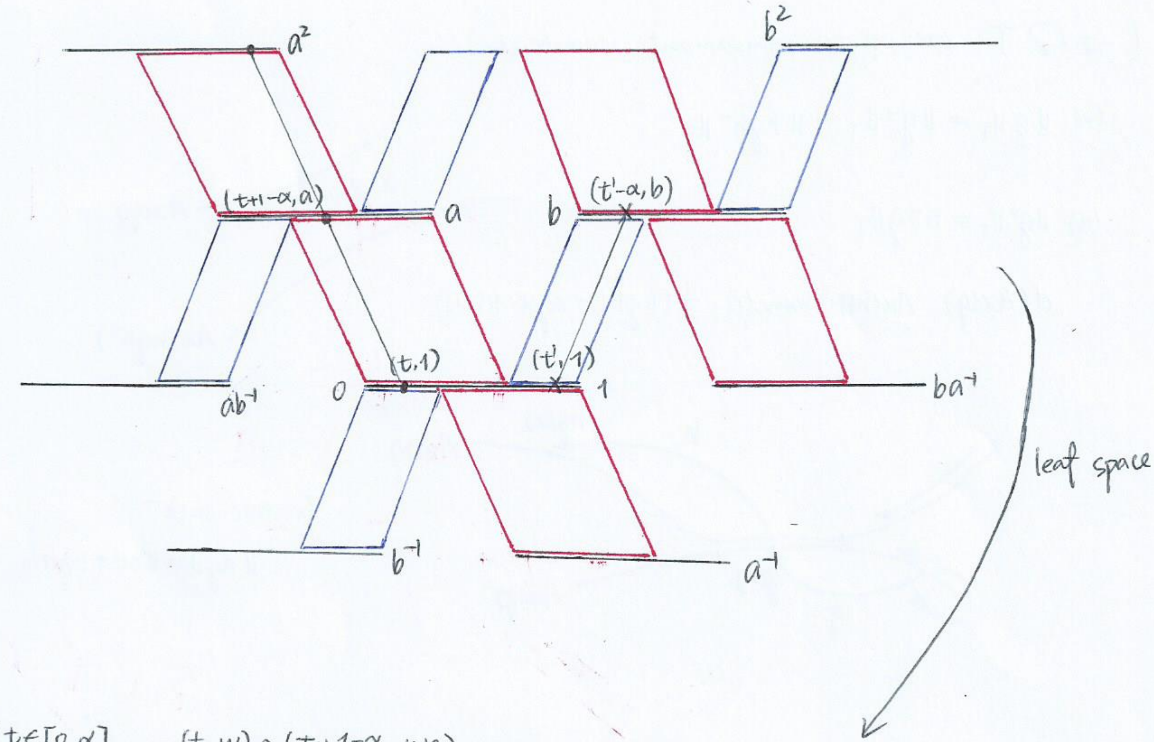
S:



"vertical foliation"

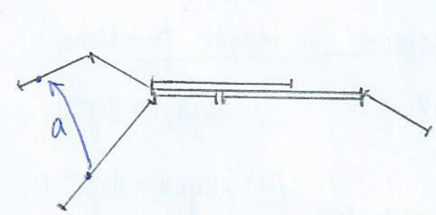
$\pi_1(S) = F_2$

$\tilde{S}$:



$\forall t \in [0,\alpha]$ $\quad (t,w) \sim (t+1-\alpha, wa)$

$\quad\quad\quad\quad\quad (t,w) \sim (t-\alpha \ \ , wb)$



$[0,1] \times F_2 / \sim$ $\quad$ action of F_2
by left multiplication
by isometries

Branch points are dense $\quad \alpha \in \mathbb{R} \setminus \mathbb{Q}$

$\|a\|_T = 1-\alpha$, $\quad \|b\|_T = \alpha \quad \|\underset{a^{-1}b^{-1}ab}{\underset{\shortparallel}{[a,b]}}\|_T = 0$

Exercise: find the fix point?

Group action on trees: $\quad$ by isometries $\quad G \xrightarrow{\alpha} \mathrm{Isom}(T)$

minimal: no G-invariant subtrees

$$G \longrightarrow \mathbb{R}_{\geq 0}$$
$$g \longmapsto \|g\|_T$$

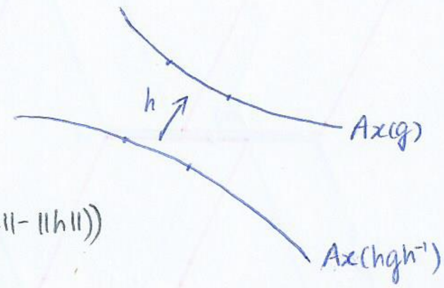
Thm [CULLER-MORGAN] The translation length functions determines T.
(except abelian actions)

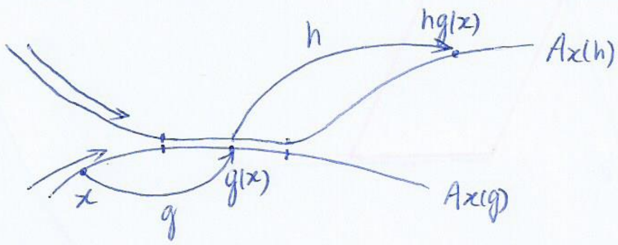
($G \curvearrowright T$ are up to equivariant isometries.)

(i) $\|g\|_T = \|g^{-1}\|_T = \|hgh^{-1}\|_T$

(ii) $\|g^n\|_T = n\|g\|_T$

$d(Ax(g), Ax(h)) = \max(0 ; \frac{1}{2}(\|gh\| - \|g\| - \|h\|))$





In this case:

$\|hg\| = \|h\| + \|g\|$

<u>Axioms for length functions</u>

$\forall\, g, h$ (i) $\|g\|_T = \|g^{-1}\|_T = \|hgh^{-1}\|_T$

closed conditions (ii) $\begin{cases} \|gh\| = \|gh^{-1}\| \\ \text{or} \\ \max(\|gh\|, \|gh^{-1}\|) \leq \|g\| + \|h\| \end{cases}$

(iii) $\|g\| > 0,\ \|h\| > 0 \implies \begin{cases} \|gh\| = \|gh^{-1}\| > \|g\| + \|h\| \\ \text{or} \\ \max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\| \end{cases}$

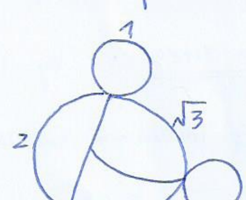
<u>Exercise</u>: proving "$\|g^n\|_T = n\|g\|_T$" using the axioms.

Outer space: $CV_N := \{\text{actions } F_N \curvearrowright T,\ T \text{ simplicial metric tree}\}$

action simplicial, minimal and free.

$CV_N \hookrightarrow \mathbb{R}^{F_N}_{\geq 0}$

$F_N \curvearrowright T \longmapsto \|\cdot\|_T$

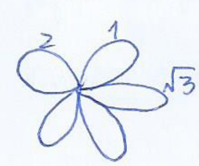


$T/F_N \simeq P$

simplicial actions

$\pi_1(P) = F_N$

free action



with length on edges

pointwise convergence topology on length functions

$\overline{CV_N}$: closure in $\mathbb{R}_{\geq 0}^{F_N}$

$\overline{CV_N}$ consists of actions on $\mathbb{R}$-trees.

$Out(F_N)$ acts on CV_N:

$$\|\cdot\|_T : F_N \longrightarrow \mathbb{R}_{\geq 0}$$

$\varphi \in Aut(F_N)$, $\|u\|_{T\circ\varphi} := \|\varphi(u)\|_T$

$\alpha : F_N \longrightarrow Isom(T)$

$\alpha \circ \varphi$

$Inn(F_N)$ acts trivially

$in : F_N \longrightarrow F_N$

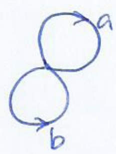
$g \longmapsto hgh^{-1}$

$\|g\|_T = \|hgh^{-1}\|_T = \|g\|_{T\circ in}$

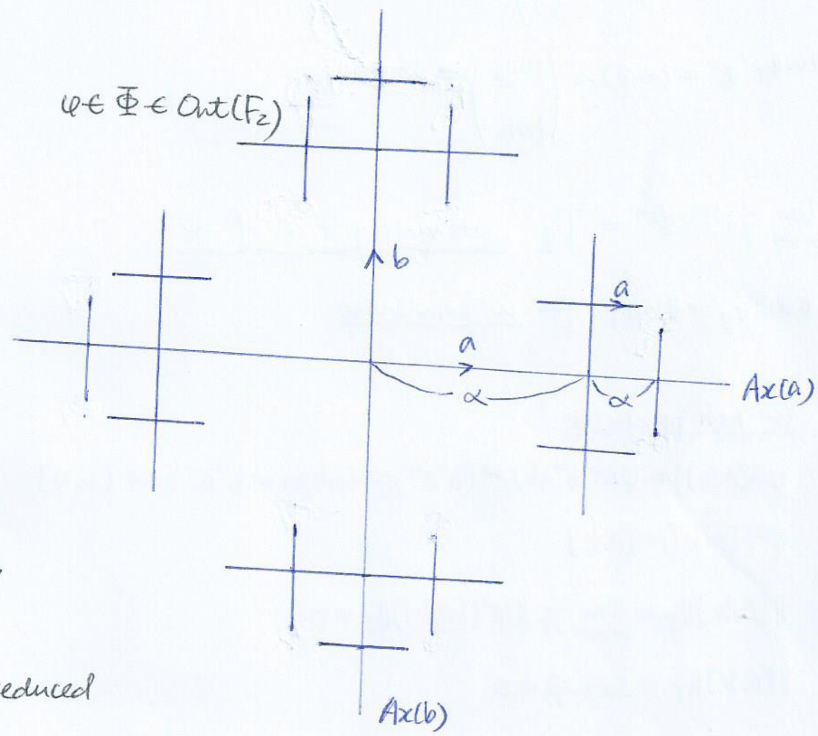
<u>Examples:</u>

$\varphi :\ a \longmapsto ab$

$\quad\ \ b \longmapsto a$



$\varphi \in \Phi \in Out(F_2)$



$l(a) = \alpha$, $l(b) = \beta$

$\|a\|_T = \alpha$, $\|b\|_T = \beta$

$w \in F_2$: cyclically reduced

$w = aba^{-1}b^{-1}$ $\quad 1 \in Ax(w)$

$\leftarrow$ number of b's in w

$\|w\|_T = \alpha|w|_a + \beta|w|_b = 2\alpha + 2\beta$

$\uparrow$

number of a's in w

$T \circ \Phi$: $\quad \|a\|_{T\circ\Phi} = \|\varphi(a)\|_T = \|ab\|_T = \alpha + \beta$

$\|b\|_{T\circ\Phi} = \|\varphi(b)\|_T = \|a\|_T = \alpha$

$\|w\|_{T\circ\Phi} = \|\varphi(w)\|_T = (\alpha, \beta)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |w|_a \\ |w|_b \end{pmatrix}$

w: pointwise word (no inverses i.e. no a^{-1} or b^{-1})

$$\|a\|_{T\circ\Phi^n} = \|\varphi^n(a)\|_T = (\alpha,\beta)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1\\0\end{pmatrix}$$

$$\|b\|_{T\circ\Phi^n} = \|\varphi^n(b)\|_T = (\alpha,\beta)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}0\\1\end{pmatrix}$$

w pointwise

$$\|w\|_{T\circ\Phi^n} = \|\varphi^n(w)\|_T = (\alpha,\beta)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}|w|_a\\|w|_b\end{pmatrix}$$

$\lambda = \frac{1+\sqrt{5}}{2}$, $\alpha = \frac{\sqrt{5}-1}{2}$, $\beta = \frac{3-\sqrt{5}}{2}$ $(\alpha+\beta)=1$

$$\|a\|_{T\circ\Phi^n} = (\alpha,\beta)\lambda^n\begin{pmatrix}1\\0\end{pmatrix} = \lambda^n\alpha$$

$$\|w\|_{T\circ\Phi^n} = (\alpha,\beta)\lambda^n\begin{pmatrix}|w|_a\\|w|_b\end{pmatrix} = \lambda^n\|w\|_T$$

$\lim_{n\to\infty}\frac{1}{\lambda^n}T\circ\Phi^n = T_\Phi$ <u>attracting tree of Φ</u>

$\|w\|_{T_\Phi} = \|w\|_T$ for <u>w pointwise</u>

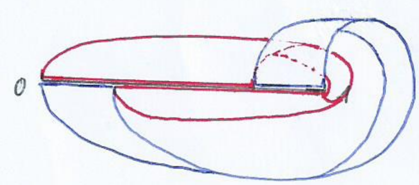
<u>w non pointwise</u>

$\varphi([a,b]) = \varphi(a^{-1}b^{-1}ab) = (b^{-1}a^{-1})a^{-1}(ab)a = b^{-1}a^{-1}ba = [b,a]$

$\varphi^2([a,b]) = [a,b]$

$\|[a,b]\|_T = \lim_{n\to\infty}\frac{1}{\lambda^n}\|\varphi^n([a,b])\|_T = 0$

$\|[a,b]\|_T = 2\alpha+2\beta \neq 0$

<u>Claim:</u> T_Φ is 

$\|a\|$, $\|b\|$ and $\|[a,b]\|$ are equal.

<u>Thm [COHEN-LUSTIG, BESTVINA-FEIGHN]</u>

$\overline{CV}_N = \{F_N \curvearrowright T$ minimal by isometries, very small $\}$

Small: (i) stablizers of non trivial arcs are cyclic

very small: (ii) $\mathrm{Fix}(g) = \mathrm{Fix}(g^n)$

(iii) $\mathrm{Fix}(g)$ doesn't contain tripods.

CULLER-MORGAN

[BF] $T \in \overline{CV}_N$

very small can be approximated by simplicial.

[CL] def.

very small + simplicial

very small + very small closed.

Remark: Free actions are very small.

Proof: 1) very small is a closed conciliation on length functions.

2) Start with a very small action

Find a sequence of approximating simplicial free actions.

System of isometries

Thm[Rips, GLP, BF]

G free group, $G \curvearrowright T$ by isometries, free
then G is a free product of $\mathbb{Z}^N$, $\pi_1(S_g)$, F_N

recall: Thm[Ibara, Bass-Serre] G free group, $G \curvearrowright T$ freely simplicial
Then G is free.

$K \subseteq T$ subtree

a restrict to a partial isometry of K. $K \cap aK \longrightarrow K \cap a^{-1}K$
$x \longmapsto xa := a^{-1}x$

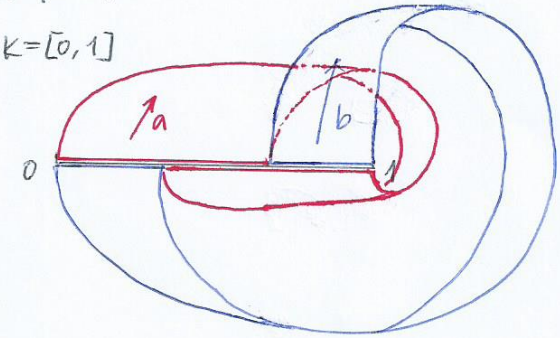
$S = (K, A)$

K big enough for the partial isometry is not empty.

$T_S = K \times F_A / \sim$ (Free group on A)

$A = \{a, b\}$

$K = [0, 1]$



$(x, w) \sim (xa, wa)$

$T_A \curvearrowright T_S$ by left multiply on 2nd coordinate by isometries.

T_S is an approximation of T,

$\forall w \in F_N$, $\|w\|_T = \|w\|_{T_S}$ if $x \in K$. $\forall u$ prefix of $x \cdot u$ in K.

Compose and inverse partial isometries:

Let $K_n \xrightarrow[n\to\infty]{} T$

$\forall w$, for n big enough, $\exists x \in K_n$ such that $x, w \in K$
and thus $\|w\|_{T_{S_n}} = \|w\|_T$, $T_{S_n} \xrightarrow[n\to\infty]{} T$

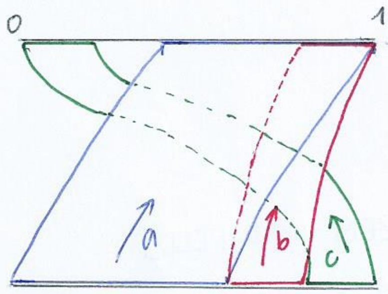
$(x,\mu) \sim (xa, \mu a) \overset{?}{\Longrightarrow}$ $H'(x,\mu) \sim H'(xa, \mu a)$

$H'(x,\mu) \sim H'(xb, \mu b)$

$H'(x,\mu) \sim H'(xc, \mu c)$

$H'(x,\mu) = (H'(x), \varphi(\mu))$

$H'(xa,\mu a) = (H'(xa), \varphi(\mu a))$



$H'(xb) = H'(x)\underbrace{caaa}_{\varphi(b)}$

H' is a homotopy of ratio $\frac{1}{\lambda}$ defined on T_S such that $\forall x \in T_S$, $\forall \mu \in F_N$

$$\underline{H'(\mu x) = \varphi(\mu) H'(x)}$$

$H = H'^{-1}$ ratio $\lambda > 1$ $\quad \underline{H(\mu x) = \varphi^{-1}(\mu) H(x)}$

$\varphi \in \Phi \in Out(F_3)$, attracting tree T_Φ, $\|\mu\|_{T_{\Phi^{-1}}} = \lim_{n\to\infty} \frac{\|\Phi^n(\mu)\|_A}{\lambda^n}$ ← cyclically reduced length

Thm [LEVITT-LUSTIG]

Any iwip automorphism acts with NORTH-SOUTH Dynamics on $\overline{PCV_N}$

$T_\psi \circ \psi = \lambda_\psi T_\psi$

$\frac{1}{\lambda_\psi} T_\psi \circ \psi = T_\psi$

$H: T_\psi \xrightarrow[\text{equivariant}]{\text{isometry}} \frac{1}{\lambda} T_\psi \circ \psi$

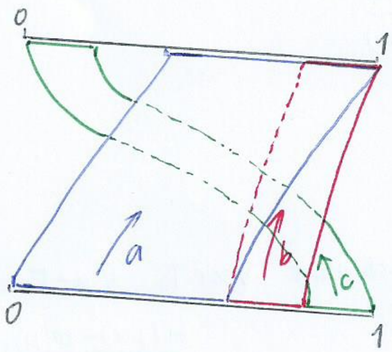
$$\frac{d(H(x), H(y))}{\lambda_\psi} = d(x,y)$$

$\underline{H(\mu \cdot x) = \psi(\mu) \cdot H(x)}$ $\quad$ Conclusion: $T_S = T_{\Phi^{-1}}$

Def: A tree is geodesic if there exists a finite K such $T_K = T$.
(T is a tree with an action of the free group.)

[SKORA] geodesic: transverse to a foliation on a surface.
Surface tree

$\underline{N=2}$: $Out(F_2) = GL_2(\mathbb{Z}) = MCG(\;)$



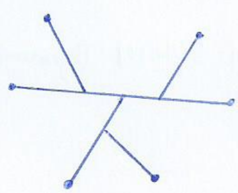
[BOSHERNITZEN-KORNFELD]

Not a surface. ([BRUIJN, TRONBETZKOY])

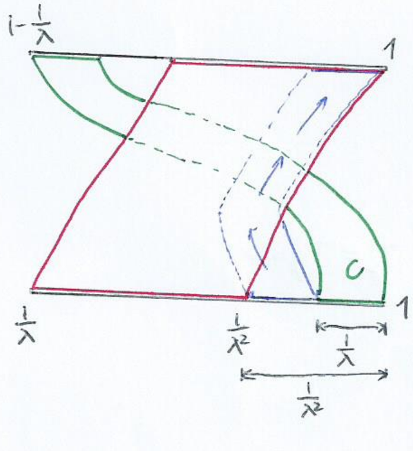
T_S geodesic

$\lambda^3 = 3\lambda^2 + \lambda - 1$, $\lambda > 1$

finite tree: convex hull of finitely many points.



any interval is a finite tree.



$$H' : [0,1] \longrightarrow [1-\tfrac{1}{\lambda}, 1]$$
$$x \longmapsto 1 - \tfrac{1}{\lambda}(1-x)$$

$H'(1) = 1$

$H'(0) = 1 - \frac{1}{\lambda}$ ratio $\frac{1}{\lambda}$

Define H' on T_S, $T_S = [0,1] \times F_N / \sim$

$H'(x, u) = (H'(x), \varphi(u))$

where φ: $a \longmapsto a$ $\quad \varphi \in Aut(F_3)$
$b \longmapsto caaa$
$c \longmapsto caa$

$\varphi \in \Phi \in Out(F_3)$

Botany of trees

$T \in \overline{CV_N}$. $F_N \curvearrowright T$ minimal, very small by isometry

T is geometric: $\exists K \subseteq T$, K finite tree such that $S = (K, A)$, $T = T_S$

system of isometries

How to recognize a geometric tree?

geometric index: $F_N \curvearrowright T$

$$\forall x \in T, \quad i_{geom}(x) = \# \left(\pi_0(T \setminus \{x\}) / Stab(x) \right) + 2rank(Stab(x)) - 2$$

$\forall x, y \in T$, $\forall \mu \in F_N$ s.t. $\mu x = y$, then $i_{geom}(x) = i_{geom}(y)$

$$i_{geom}(T) = \sum_{[x] \in T/F_N} i_{geom}(x)$$

Thm [GABORIAN - LEVITT] $T \in \overline{CV_N}$

① $i_{geom}(T) \leq 2N-2$

② $i_{geom}(T) = 2N-2 \iff T$ is geodesic

Thm [Coulbois - Hilion - Lustig]

A basis of F_N. as points in $\overline{CV_N}$

$\forall T \in \overline{CV_N}$, $\exists K \subseteq \overline{T}$ compact, $S = (K, A)$, such that $\begin{cases} T = T_K \\ \text{or} \quad \overline{T} = \overline{T}_K \\ \text{or} \quad T = T_K^{min} \end{cases}$ (length functions are the same)

metric complication of T

To be Levitt or not?

The map Q:

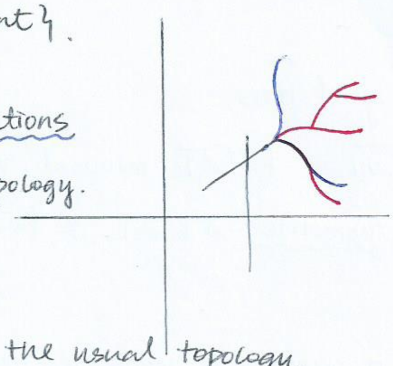
Thm: $T \in \overline{CV_N}$ with dense orbits.

$\exists!\ Q: \partial F_N \longrightarrow \hat{T}^{obs} = T \cup \partial T$ (with the observious topology) equivariant and continuous.

$\forall l \in T$, Q extends the orbit map: $F_N \longrightarrow T$

$\mu \longmapsto \mu l$

directions: connected component of $T \setminus \{point\}$.

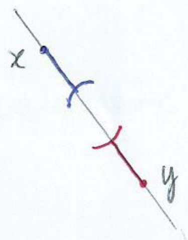


basic open sets of the observious topology: directions

observious topology is weaker than $\mathbb{R}$-trees topology.

T^{obs} Hausdorff.

induced topology on a segment coinside with the usual topology.



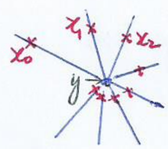
connected subspaces of T^{obs} are exactly the same as for the usual topology

$\Rightarrow \forall x, y \in T^{obs}$, $\exists!$ continuous injective path.

T^{obs} is a dendrite. [BOWDITCH]

metrizable to an $\mathbb{R}$-tree.

T^{obs} is compact.



$x_n \xrightarrow[n\to\infty]{} y$ for the observious topology.

$Q: \partial F_N \longrightarrow \hat{T}^{obs}$ continuous, equivariant.

$L(T) = \{(X,Y) \in \partial^2 F_N \mid Q(X) = Q(Y)\}$ dual lamination of T

$\partial^2 F_N = \partial F_N \times \partial F_N \setminus$ diagonal

$(L(T_\Phi) = \Lambda_{\Phi^{-1}})$

Q-index of T: $T \in \overline{CV_N}$, $F_N \curvearrowright T$ free, with dense orbits.

$\forall P \in \hat{T}$, $i_Q(P) = \# Q^{-1}(P) - 2$

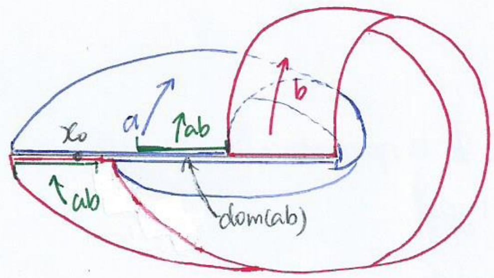
$$i_Q(T) = \sum_{[P] \in T/F_N} \max(0; i_Q(P))$$

Thm [Coulbois-Hilion]

① $i_Q(T) \leq 2N-2$

② $i_Q(T) = 2N-2 \iff T$ is of surface type.

$T =$



All points have Q-index ≥ 2
$= 2$
except the orbits of x_0

$i_Q(T) = 2N-2$

This tree is a surface tree.

$a^2 b^{-1}$ is empty

$S = (K, A)$ system of isometries (K compact $\mathbb{R}$-tree)

$\forall a \in A,\ a : K \longrightarrow K$ partial isometry between two closed subtrees of K, nonempty.
$x \longmapsto xa$

compose and inverse partial isometries.

for any reduced word $w \in F_A$, w is a partial isometry

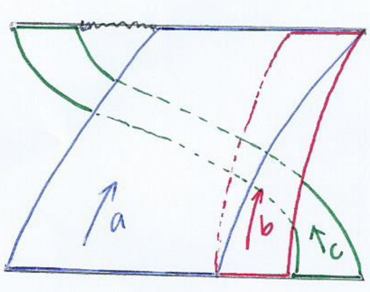
$\begin{cases} w \text{ non-empty} & \operatorname{dom} w \text{ closed subtree of } K. \\ w \text{ is empty} & \end{cases}$

w is admissible.

if $w = u \cdot v$ admissible $\Rightarrow \operatorname{dom} w \subseteq \operatorname{dom} u$

$X \in \partial F_N$ an infinite word, X is admissible if all prefixes are admissible.

$\{Q(X)\} = \bigcap_{n \geq 1} \operatorname{dom} X_n \leftarrow$ prefix of length n



$\forall P,\ i_Q(P) \geq 1$

$\left.\begin{array}{l} \Omega = \{P \in T \mid i_Q(P) \geq 2\} \\ \text{totally disconnected} \end{array}\right] \Rightarrow$ the tree is of Levitt type.

$\varphi:\ a \longmapsto b$
$b \longmapsto caaa$
$c \longmapsto caa$

$T\Phi^{-1}$ is geometric Levitt.

$T\bar{\Phi}^{-1}$ is not geometric and of surface type.

Thm [Coulbois-Hilion]

$\Phi \in Out(F_N)$ iwip

① T_Φ and $T_{\Phi^{-1}}$ are geometric $L\Phi$ is geometric

$\Longleftrightarrow$ T_Φ and $T_{\Phi^{-1}}$ are surface type.

$\Longleftrightarrow$ Φ can be realized as pseudo-Anosov Mapping Class.

$\Longleftrightarrow$ T_Φ is a surface tree.

or ② T_Φ is geometric, $T_{\Phi^{-1}}$ is not geometric $L\Phi$ is para-geometric

T_Φ is Levitt, $T_{\Phi^{-1}}$ is surface type.

or ③ $T_{\Phi^{-1}}$ is geometric, T_Φ is not geometric

$T_{\Phi^{-1}}$ is Levitt, T_Φ is surface type.

or ④ T_Φ and $T_{\Phi^{-1}}$ are not geometric and of Levitt type. $L\Phi$ pseudo-Levitt.