COUNTEREXAMPLES TO HEDETNIEMI’S CONJECTURE AND INFINITE BOOLEAN LATTICES

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Abstract. We prove that for any $c \geq 5$, there exists an infinite family $(G_n)_{n \in \mathbb{N}}$ of graphs such that $\chi(G_n) > c$ for all $n \in \mathbb{N}$ and $\chi(G_m \times G_n) \leq c$ for all $m \neq n$. These counterexamples to Hedetniemi’s conjecture show that the Boolean lattices of exponential graphs with $K_c$ as a base are infinite for $c \geq 5$.

1. Introduction

The categorical product of two graphs $G$ and $H$ is the graph $G \times H$ with vertex-set $V(G \times H) = V(G) \times V(H)$, whose edges are the pairs $\{(g_1, h_1), (g_2, h_2)\}$ such that $\{g_1, g_2\}$ is an edge of $G$ and $\{h_1, h_2\}$ is an edge of $H$. Hedetniemi’s conjecture of 1966 [12] states that the chromatic number of a categorical product of graphs is equal to the minimum of the chromatic numbers of the factors. In 2019, Shitov [14] refuted the conjecture by constructing counterexamples for very large chromatic numbers.

Shitov’s construction was subsequently adapted and modified by many authors. The asymptotic bounds on the difference $\min\{\chi(G), \chi(H)\} - \chi(G \times H)$ and on the ratio $\min\{\chi(G), \chi(H)\}/\chi(G \times H)$ were investigated in [20, 11, 26]; in [26] it is shown that the ratio $\min\{\chi(G), \chi(H)\}/\chi(G \times H)$ can get arbitrarily close to 2.

In another direction, the sizes and chromatic numbers of counterexamples were gradually decreased in [27, 17, 23]. By now it is known that counterexamples to Hedetniemi’s conjecture exist for any chromatic number at least 5.

Here, we prove the following.

Theorem 1. For any $c \geq 5$, there exists an infinite family $(G_n)_{n \in \mathbb{N}}$ of graphs such that $\chi(G_n) > c$ for all $n \in \mathbb{N}$ and $\chi(G_m \times G_n) \leq c$ for all $m \neq n$.

The reason to concoct this new style of disproof of Hedetniemi’s conjecture is connected to the reason why Hedetniemi’s conjecture was appealing in 1966. The identity

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

remains valid in many cases. In particular, it is not hard to show that a categorical product of nonbipartite graphs remains nonbipartite, by identifying an odd cycle in a categorical product of odd cycles. The structure of the critical graphs for higher chromatic numbers is not as well understood. However in 1966, Hajós’ construction looked promising, and NP-completeness had not yet been formulated. It was reasonable to hope that the structure of the critical graphs might become sufficiently clear, and that the odd cycle argument could be adapted to higher chromatic numbers.

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But things did not turn out that way. Instead of a general understanding of critical graphs, various lower bounds on the chromatic number have been devised over time. Among these, we find topological bounds, fractional chromatic numbers and much more. Each bound can be tight or not, depending on the class of graphs considered. Now, adaptations of Hedetniemi’s conjecture can be formulated for each lower bound on the chromatic number. Indeed many of these have been proved over the years (see [25, 16, 8, 1]). However, the list of useful lower bounds on the chromatic number is not exhausted; where will others be found?

The natural context of Theorem 1 is that of the Boolean exponential lattices $K^G_c$ that will be presented in the next section. Theorem 1 is a reformulation of the fact that $K^G_c$ is infinite for any $c \geq 5$. Any lower bound $\beta$ on the chromatic number that satisfies the identity $\beta(G \times H) = \min\{\beta(G), \beta(H)\}$ corresponds to a filter in $K^G_c$. Therefore, understanding the structure of $K^G_c$ may be relevant.

2. Exponential Graphs

For graphs $G$ and $H$, we write $G \to H$ if there exists a homomorphism from $G$ to $H$, and $G \leftrightarrow H$ if $G \to H$ and $H \to G$. Let $\mathcal{G}$ be the class of finite graphs. The relation $\to$ is transitive and its quotient by the equivalence $\leftrightarrow$ gives rise to a distributive lattice order on $\mathcal{G}$, with $\times$ as meet and the disjoint union $+$ as join.

The chromatic number of $G$ is the least integer $c$ such that $G \to K_c$, where $K_c$ is the complete graph on the vertex-set $\{1, \ldots, c\}$. Hedetniemi’s conjecture states that every-complete graph is meet-irreducible in $\mathcal{G}/\leftrightarrow$. Indeed if $G \times H \to K_c$, then $K_c \leftrightarrow (G + K_c) \times (H + K_c)$; meet-irreducibility then implies $G + K_c \to K_c$ or $H + K_c \to K_c$.

For graphs $K$, $G$, the exponential graph $K^G$ is the graph whose vertices are the functions from the vertex-set of $G$ to that of $K$; and whose edges are the pairs $(f_1, f_2)$ of functions such that for every edge $(g_1, g_2)$ of $G$, $(f_1(g_1), f_2(g_2))$ is an edge of $K$. The properties of exponentiation are well-known, and exposed in [3, 9]. We list some of the most relevant properties here.

(i) $G \times H \to K$ if and only if $H \to K^G$. In particular, $K^G$ contains a loop if and only if $G \to K$.
(ii) If $G \to H$, then $K^H \to K^G$.
(iii) $K^{K^G} \leftrightarrow K^G$
(iv) $K^{G+H} \leftrightarrow K^G \times K^H$.
(v) The constant maps induce a copy of $K$ in $K^G$.

In particular, the identity $G \times K^G_c \to K_c$ always holds. Hence Hedetniemi’s conjecture is equivalent to the statement that if $\chi(G) > c$, then $\chi(K^G_c) \leq c$ (see [7]).

We denote $K^G$ the class of exponential graphs $K^G, G \in \mathcal{G}$, quotiented by $\leftrightarrow$. Its minimal element is $K$, and its maximal element is the single vertex with a loop, denoted $1$. (Note that we commit an abuse of notation and talk of graphs as elements of $K^G$ even though formally, the elements of $K^G$ are equivalence classes of graphs.) As an ordered set, $K^G$ inherits the meet $\times$ from $\mathcal{G}$, though it is not a sublattice of $\mathcal{G}$ because it is not closed under disjoint unions. However the identity $K^{G+H} \leftrightarrow K^G \times K^H$ allows to define a join on $K^G$. It is well known (see [3]) that $K_c^G$ is a Boolean lattice for any $K$. Hedetniemi’s conjecture states that for any $c$, $K^G_c$ is the two-element lattice $\{K_c, 1\}$. Theorem 1 is equivalent to the statement that for $c \geq 5$, $K^G_c$ is an infinite Boolean lattice (see [19]).
Theorem 1 will be proved by constructing an antichain from a chain in $K_n^c$. We start with $H_0$ such that $\chi(H_0) > c$ and $\chi(K_h^{H_0}) > c$. It is easy to find an infinite sequence $(H_n)_{n \in \mathbb{N}}$ such that $H_{n+1} \rightarrow H_n$ and $\chi(H_n) > c$ for all $n \in \mathbb{N}$. We then have $K_c^{H_n} \rightarrow K_c^{H_{n+1}}$, so that $\chi(K_c^{H_n}) > c$ for all $n \in \mathbb{N}$. Consider the sequence $(G_n)_{n \in \mathbb{N}}$ defined by $G_n = H_n \times K_c^{H_{n+1}}$. It is not hard to show that $(G_n)_{n \in \mathbb{N}}$ satisfies the conclusion of Theorem 1. Indeed for $m > n$ we have
\[ G_m \times G_n \rightarrow H_m \times K_c^{H_{n+1}} \rightarrow H_{n+1} \times K_c^{H_{n+1}} \rightarrow K_c. \]

The difficulty is in proving $\chi(G_n) > c$, that is, $K_c^{H_{n+1}} \not\rightarrow K_c^{H_n}$. Even though it is easy to devise $(H_n)_{n \in \mathbb{N}}$ such that $H_n \not\rightarrow H_{n+1}$, the difficulty lies in having also $K_c^{H_{n+1}} \not\rightarrow K_c^{H_n}$, or equivalently $K_c^{K_c^{H_n}} \not\rightarrow K_c^{K_c^{H_{n+1}}}$. Our sequence $(H_n)_{n \in \mathbb{N}}$ will consist of some “universal graphs for wide colourings” presented in the next section.

3. Wide colourings

For a graph $G$ and an odd integer $w = 2v + 1$, the graph $\Gamma_w(G)$ has the same vertices as $G$, and two vertices are connected by an edge in $\Gamma_w(G)$ if they are endpoints of a walk of length $w$ in $G$. Thus $\Gamma_w(G)$ contains loops only if the odd girth of $G$ is at most $w$. Otherwise, $\Gamma_w(G)$ admits a proper vertex-colouring with sufficiently many colours. A proper colouring of $\Gamma_w(G)$ is called a $(v + 1)$-wide colouring of $G$. (Here and below, we will use $v$ for $\lfloor w/2 \rfloor$.)

The functor $\Gamma_w$ has a right adjoint $\Omega_w$, which we describe next. For a graph $H$, $\Omega_w(H)$ is the graph whose vertices are the sequences $(X_0, X_1, \ldots, X_v)$ of nonempty sets of vertices of $H$ satisfying the following properties.

1. $X_0$ is a singleton $\{x\}$,
2. for $i \in \{1, \ldots, v\}$, every vertex of $X_{i-1}$ is connected by an edge to every vertex of $X_i$,
3. for $i \in \{1, \ldots, v - 1\}$, $X_{i-1} \subseteq X_{i+1}$.

The edges of $\Omega_w(H)$ join the pairs $(X_0, \ldots, X_v), (Y_0, \ldots, Y_v)$ satisfying the following properties.

1. For $i \in \{1, \ldots, v\}$, $X_{i-1} \subseteq Y_i$,
2. every vertex of $X_v$ is connected by an edge to every vertex of $Y_v$.

Lemma 2 ([10]). For two graphs $G, H$, we have $\Gamma_w(G) \rightarrow H$ if and only if $G \rightarrow \Omega_w(H)$.

Thus a graph $G$ admits a $(v + 1)$-wide colouring with $c$ colours if and only if $\Gamma_w(G) \rightarrow K_c$ that is, $G \rightarrow \Omega_w(K_c)$. Thus the graphs $\Omega_w(K_c)$ are the “Universal graphs for wide colourings”. Note that their construction resembles that of Kneser graphs: adjacency in properties (ii) and (iv) above is disjointness. As explained in [6], the existence of graphs that admit optimal colourings that are wide was conjectural at some point. This question is now settled:

Lemma 3 ([10, 22]). For $c, v \geq 0$ and $w = 2v + 1$, $\chi(\Omega_w(K_c)) = c$.

In particular, $\Omega_w(K_c)$ is a $c$-chromatic graph which admits a $(v + 1)$-wide colouring with $c$ colours. (Note that the result is also implicit in [6, 2, 15], with alternative presentations of $\Omega_w(K_c)$.)
We will use the fact that $\Omega_w(K_c)$ is connected and has odd girth $w$. This is not hard to prove, either through the explicit description of $\Omega_w(K_c)$ or through an appropriate use of Lemma 2. Thus, while we have $\Omega_{w+2}(K_c) \rightarrow \Omega_w(K_c)$ for any odd $w$, we also have $\Omega_w(K_c) \not\rightarrow \Omega_{w+2}(K_c)$. We use two further folklore properties of these functors. We include a proof for convenience.

**Lemma 4.** Let $a, b$ be odd integers. Then

(i) $\Omega_a(G \times H) \leftrightarrow \Omega_a(G) \times \Omega_a(H)$,

(ii) $\Omega_a(\Gamma_b(H)) \leftrightarrow \Omega_{ab}(H)$.

**Proof.** Property (i) folds for any right adjoint: we have

$K \rightarrow \Omega_a(G \times H) \leftrightarrow \Gamma_a(K) \rightarrow G \times H \rightarrow G$ (resp. $H$)

$\Leftrightarrow K \rightarrow \Omega_a(G)$ (resp. $\Omega_a(H)$)

$\Leftrightarrow K \rightarrow \Omega_a(G) \times \Omega_a(H)$.

Property (ii) follows from the identity $\Gamma_b(\Gamma_a(G)) = \Gamma_{ab}(G)$, which is obvious from the definition:

$G \rightarrow \Omega_{ab}(H) \leftrightarrow \Gamma_b(\Gamma_a(G)) = \Gamma_{ab}(G) \rightarrow H$

$\Leftrightarrow \Gamma_a(G) \rightarrow \Omega_b(H)$

$\Leftrightarrow G \rightarrow \Omega_a(\Omega_b(H))$.

□

4. **Main results and open problems**

The graphs $\Omega_w(K_c)$ are used in the construction of counterexamples to Hedetniemi’s conjecture in [17, 23]. In particular, Wrochna proved in [23] that for $c \geq 5$, $\chi(K_c^{\Omega_{13+2c}}) > c$. In the next section, we adapt his arguments to prove our main auxiliary result:

**Lemma 5.** For $c \geq 5$ and $w \geq 7$,

$K_cK_c^{\Omega_{2w-1}(K_{2c-2})} \rightarrow K_c + \Omega_w(K_{2c-2})$.

Note that $K_c + \Omega_w(K_{2c-2}) \rightarrow K_c^{\Omega_{w}(K_{2c-2})}$, so that Lemma 5 implies that $K_c + \Omega_w(K_{2c-2})$ is sandwiched between two nontrivial elements of $K_c^{\nu}$. It would be desirable to prove a stronger result such as $K_c^{\Omega_{w}(K_{2c-2})} \leftrightarrow K_c + \Omega_w(K_{2c-2})$, for large enough values of $w$. We first show how Lemma 5 is used to prove Theorem 1.

**Proof of Theorem 1.** Let $\omega(0) = 5$ and $\omega(k) = 2\omega(k-1) + 3$ for $k \geq 0$. By Lemma 5, we have

$K_c^{\Omega_{\omega(k)(K_{2c-2})}} \rightarrow K_c + \Omega_{\omega(k-1)+2}(K_{2c-2})$.

Therefore $\Omega_{\omega(k-1)(K_{2c-2})} \not\rightarrow K_c^{\Omega_{\omega(k)(K_{2c-2})}}$, since $\Omega_{\omega(k-1)(K_{2c-2})}$ is connected, $(2c-2)$-chromatic and has odd girth $\omega(k-1)$. This implies

$\chi\left(\Omega_{\omega(k-1)(K_{2c-2})} \times K_c^{\Omega_{\omega(k)(K_{2c-2})}}\right) > c$

for all $k \geq 1$. Hence the sequence $(G_k)_{k \in \mathbb{N}}$ of graphs defined by

$G_k = \Omega_{\omega(k)(K_{2c-2})} \times K_c^{\Omega_{\omega(k+1)(K_{2c-2})}}$
satisfies \( \chi(G_k) > c \) for all \( k \), and for \( i > j \),
\[
G_i \times G_j \rightarrow \Omega_{\omega(i)}(K_{2c-2}) \times K_c^{\Omega_{\omega(j+1)}(K_{2c-2})} \rightarrow \Omega_{\omega(j+1)}(K_{2c-2}) \times K_c^{\Omega_{\omega(j+1)}(K_{2c-2})} \rightarrow K_c.
\]

By Theorem 1, the Boolean lattice \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \) is infinite for all \( c \geq 5 \). Many questions remain about the structure of this lattice. In particular, the \textit{weak Hedetniemi conjecture} states that for every chromatic number \( c \), there exists a bound \( b(c) \) such that if \( G, H \) satisfy \( \chi(G) \geq \chi(H) \geq b(c) \), then \( \chi(G \times H) > c \). It is known (see [24]) that the weak Hedetniemi conjecture is equivalent to the statement that \( b(9) \) is well-defined. If the weak Hedetniemi conjecture is false, then for \( c \geq 9 \), \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \) is a dense Boolean lattice containing graphs with arbitrarily large chromatic numbers (see [18]). On the other hand, if for some \( c \geq 9 \), \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \) contains at least one atom, then the weak Hedetniemi conjecture is true. Here we note that the strengthening \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \leftrightarrow K_c + \Omega_{\omega(9)}(K_{2c-2}) \) of Lemma 5 would give a conclusion consistent with the fallacy of the weak Hedetniemi conjecture.

**Proposition 6.** Suppose that for all \( c \geq 5 \), there exists an odd \( w(c) \geq 13 \) such that for all odd \( w \geq w(c) \),
\[
K_c^{\Omega_{\omega(9)}(K_{2c-2})} \leftrightarrow K_c + \Omega_{\omega(9)}(K_{2c-2}).
\]

Then for all \( c \geq 5 \), the Boolean lattice \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \) is dense and contains graphs with arbitrarily large chromatic numbers.

**Proof.** For odd \( w \geq 13 \) and \( c \geq 5 \), let \( d(w, c) \) be the supremum of integers \( d \) satisfying \( \chi(K_c^{\Omega_{\omega(9)}(K_{2c-2})}) > c \). (That is, \( d(w, c) \) is either a maximum, or it is infinite.) For any odd \( w < w' \), we have \( d(w, c) \leq d(w', c) \); in particular, \( d(w, c) \geq d(13, c) \geq 2c-2 \).

More precisely, Wrochna’s method [23] proves that in \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \), the connected component containing the constant maps already has chromatic number larger than \( c \). We will prove that when \( c \) is fixed, \( d(w, c) \) goes to infinity as \( w \) increases.

Given \( c \geq 5 \), \( w \geq 13 \), let \( d \) be any integer at most \( d(2w, 2c-2) \). Then from
\[
\Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \rightarrow K_{2c-2},
\]
we fix \( w' \geq w(c) \) and apply the functor \( \Omega_{\omega(9)} \), using the rules \( \Omega_{\omega(9)}(G \times H) \leftrightarrow \Omega_{\omega(9)}(G) \times \Omega_{\omega(9)}(H) \) and \( \Omega_{\omega(9)}(\omega_b(G)) \leftrightarrow \omega_b(G) \) from Lemma 4. We get
\[
\Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \rightarrow \Omega_{\omega(9)}(K_{2c-2}).
\]

Multiplying both sides by \( K_c^{\Omega_{\omega(9)}(K_{2c-2})} \) gives
\[
\Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \rightarrow \Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \rightarrow K_c.
\]

Therefore
\[
\Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \rightarrow K_c^{\Omega_{\omega(9)}(K_{2c-2})}.
\]

We will show that
\[
\Omega_{\omega(9)}(K_{2c-2}) \times K_c^{\Omega_{\omega(9)}(K_{2c-2})} \not\leftrightarrow K_c.
\]

Suppose for a contradiction that this is not the case. Then
\[
\Omega_{\omega(9)}(K_{2c-2}) \rightarrow K_c^{\Omega_{\omega(9)}(K_{2c-2})} \leftrightarrow K_c + \Omega_{\omega(9)}(K_{2c-2}).
\]
Let $H$ be the connected component of $K_{2c-2}^{\Omega_w(K_d)}$ containing the constant maps. Then $\chi(H) > 2c-2$, and $H$ contains a complete subgraph of size $2c-2$ induced by the constant maps. Therefore $\chi(\Omega_w'(H)) \geq \chi(\Omega_{w'}'(K_{2c-2})) = 2c-2$. This implies $\Omega_{w'}(H) \nrightarrow K_c$, hence $\Omega_{w'}(H) \rightarrow \Omega_{w'}(K_{2c-2})$. Applying the functor $\Gamma_{w'}$, we then get
\[ H \leftrightarrow \Gamma_{w'}(\Omega_{w'}(H)) \rightarrow \Gamma_{w'}(\Omega_{w'}(K_{2c-2})) \leftrightarrow K_{2c-2}. \]
This is a contradiction since $\chi(H) > 2c-2$.

Therefore
\[ \chi\left(K_c^{\Omega_{w'}(K_d)}\right) \geq \chi\left(\Omega_{w'}(K_{2c-2}) \times K_c^{\Omega_{w'}(K_{2c-2})}\right) > c. \]

We have shown that $d(wu',v) \geq d(w,2c-2)$. Since we have $d(w,v) \geq 2c-2$ for all values $w,v$, this implies that for fixed $c$, $d(w,v)$ goes to infinity as $w$ increases. We now fix a value of $c$ and examine the properties of $K^G_c$ in light of the above. For any $d$, there exists $w$ such that $\chi(K_c^{\Omega_{w'}(K_d)}) > c$. The graph $K_c^{\Omega_{w'}(K_d)}$ is then an element of $K^G_c$ with chromatic number at least $d$. The density of $K^G_c$ follows then from the standard argument dating back to Nešetřil and Perles (see [9, 18]): Let $H$ be an nonminimal element of $K^G_c$. Fix $d \geq \chi(H^c)$, and $w \geq w(c)$ such that $w$ is larger than the odd girth of any connected component of $H$ with chromatic number larger than $c$. Then $K_c^{\Omega_{w'}(K_d)} \nrightarrow K_c + \Omega_w(K_d)$ is an element of $K^G_c$. We have
\[ K_c \rightarrow H \times K_c^{\Omega_{w'}(K_d)} \rightarrow H, \]
but $H \nrightarrow H \times K_c^{\Omega_{w'}(K_d)}$ because the odd girth of some connected component of $H$ is too small, and $H \times K_c^{\Omega_{w'}(K_d)} \nrightarrow K_c$ because $\Omega_{2w+1}(K_d) \nrightarrow K_c^H$. Thus, $K^G_c$ has no atoms, and is a dense boolean lattice. □

The outstanding open problem is whether the hypothesis in Proposition 6 is true. The next section contains the proof of Lemma 5, followed by a discussion of the improvements needed to prove this hypothesis.

5. Proof of Lemma 5

To prove $K_c^{\Omega_{2w-1}(K_{2c-2})} \rightarrow K_c + \Omega_w(K_{2c-2})$, we will show that the connected components of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ that are not $c$-colourable admit a $(v+1)$-wide colouring with $2c-2$ colours. Here, $w = 2v+1$, so that $2w - 1 = 4v + 1$. Hence the elements of $\Omega_{2w-1}(K_{2c-2})$ are $(2v+1)$-tuples $(X_0,\ldots,X_{2v})$ of sets of vertices in $K_{2c-2}$. For a technical reason, it is best to represent the elements of $K_{2c-2}$ as pairs $(x,q)$, with $x \in \{1,\ldots,c-1\}$ and $q \in \{1,2\}$. The elements of $K_c$ are just the integers $1,\ldots,c$. The elements of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ will be represented by lower-case greek letters, and those of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ by lower-case roman letters. The functions on subsets of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ will be represented by upper-case greek letters; these include $c$-colourings of some components, automorphisms, and a $(v+1)$-wide colouring of some components with $2c-2$ colours.

5.1. Constant maps. We will denote $\iota_x : \Omega_{2w-1}(K_{2c-2}) \rightarrow K_c$ the constant map with constant value $x$. The set $C = \{\iota_1,\ldots,\iota_c\}$ is a complete subgraph of size $c$ in $K_c^{\Omega_{2w-1}(K_{2c-2})}$. The elements $f$ of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ fall into two categories:

(i) Those whose restriction to $C$ is bijective.
(ii) Those whose restriction to $C$ contains a repeated colour.

If the restriction of $f$ to $C$ is bijective and $g$ is adjacent to $f$, then the restriction of $g$ to $C$ coincides with that of $f$. Thus, any connected component $H$ of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ either has all its elements bijective and identical on $C$, or all its elements nonbijective on $C$.

Let $H$ be a component of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ whose elements are nonbijective on $C$. Let $\Phi : H \to K_c$ be defined by letting $\Phi(f)$ be a colour such that there are distinct values $x_f, y_f \in \{1, \ldots, c\}$ of $C$ with $f(x_f) = f(y_f)$. If $f$ and $g$ are adjacent in $H$, without loss of generality we have $x_f \neq y_g$, so $\Phi(f) = f(x_f) \neq g(y_g) = \Phi(g)$. This shows that $\Phi$ is a proper colouring, hence $H \to K_c$.

Let $H$ be a component whose elements are bijective on $C$. Let $\pi$ be the permutation of $V(K_c)$ such that $f(x) = \pi_x$ for every $f$ in $H$. Consider the automorphism $A$ of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ defined by $A(f) = \pi^{-1} \circ f$. Then $H$ is isomorphic to $A(H)$, and for every $f \in V(H)$, $A(f)$ maps every element of $C$ to its constant value.

Thus we can restrict our attention to the subgraph of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ induced by the functions that map every element of $C$ to its constant value. For any two such adjacent functions $f, g$, for every $\lambda \in K^{\Omega_{2w-1}(K_{2c-2})}_c$ and for every $x$ that is not in the image of $\lambda$, we have $f(\lambda)$ adjacent to $g(\lambda) = x$. Let $H_{\text{id}}$ be the subgraph of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ consisting of the functions that map every element of $C$ to its constant value, and moreover are not isolated. Then for every $f$ in $H_{\text{id}}$ and $\lambda \in K^{\Omega_{2w-1}(K_{2c-2})}_c$, $f(\lambda)$ is in the image of $\lambda$. We will construct a $(2v+1)$-wide colouring $\Phi : H_{\text{id}} \to K_{2c-2}$.

5.2. Elements of $K^{\Omega_{2w-1}(K_{2c-2})}_c$. Recall that the elements of $K_{2c-2}$ are denoted as pairs $(x, q)$, with $x \in \{1, \ldots, c-1\}$ and $q \in \{1, 2\}$. Let $\gamma$ be the element of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ defined by $\gamma(X_0, \ldots, X_{2v}) = x$, where $X_0 = \{(x, q)\}$. For every $f$ in $H_{\text{id}}$, $f(\gamma) \in \{1, \ldots, c-1\}$. We define $\Phi_0(f) = f(\gamma)$. The $(v+1)$-wide colouring $\Phi : H_{\text{id}} \to K_{2c-2}$ will be of the form $\Phi(f) = (\Phi_0(f), \Phi_1(f))$ for a suitably defined $\Phi_1$.

For $x \in \{1, \ldots, c-1\}$, $\ell \in \{1, \ldots, 2v-1\}$ and $i, j \in \{1, \ldots, c\}$, the two-colouring $\tau^{x,\ell}_{i,j}$ is the element of $K^{\Omega_{2w-1}(K_{2c-2})}_c$ defined by

$$\tau^{x,\ell}_{i,j}(X_0, \ldots, X_{2v}) = \begin{cases} i & \text{if } \{(x, 1), (x, 2)\} \cap X_\ell \neq \emptyset, \\ j & \text{otherwise.} \end{cases}$$

For $x \in \{1, \ldots, c-1\}$, we then have $\gamma$ adjacent to $\tau^{x,1}_{c,x}$ in $K^{\Omega_{2w-1}(K_{2c-2})}_c$. Indeed, let $(X_0, \ldots, X_{2v})$ and $(Y_0, \ldots, Y_{2v})$ be neighbours in $\Omega_{2w-1}(K_{2c-2})$. We need to show that $\gamma(X_0, \ldots, X_{2v}) \neq \tau^{x,2}_{c,x}(Y_0, \ldots, Y_{2v})$, but the only way $\gamma(X_0, \ldots, X_{2v})$ can even be in the image of $\tau^{x,1}_{c,x}$ is if $X_0 = \{(x, q)\}$ so that $\gamma(X_0, \ldots, X_{2v}) = x$. By definition of adjacency in $\Omega_{2w-1}(K_{2c-2})$ we then have $\{(x, 1), (x, 2)\} \cap Y_1 \neq \emptyset$, so that $\tau^{x,1}_{c,x}(Y_0, \ldots, Y_{2v}) = c$. Thus, $\gamma$ is adjacent to $\tau^{x,1}_{c,x}$.

Moreover, $\tau^{x,\ell}_{i,j}$ is adjacent to $\tau^{x,\ell+1}_{k,i}$ for any three distinct values $i, j, k \in \{1, \ldots, c\}$. Indeed $i$ is the only common value in the image of these two functions. If $(Y_0, \ldots, Y_{2v})$ is adjacent to $(X_0, \ldots, X_{2v})$ with $\tau^{x,\ell}_{i,j}(X_0, \ldots, X_{2v}) = i$, then

$$\{(x, 1), (x, 2)\} \cap X_\ell \subseteq \{(x, 1), (x, 2)\} \cap Y_{\ell+1} \neq \emptyset,$$

therefore $\tau^{x,\ell+1}_{k,i}(Y_0, \ldots, Y_{2v}) = k$. 
For $x \in \{1, \ldots, c-1\}$ and $j \in \{3, \ldots, c\}$, the clique-member $\kappa^x_i$ is the element of $K^{\Omega_{2w-1}(K_{2c-2})}$ defined by

$$
\kappa^x_i(X_0, \ldots, X_{2v}) = \begin{cases} 
1 & \text{if } (x, 1) \in X_{2v}, \\
2 & \text{if } (x, 1) \not\in X_{2v} \text{ and } (x, 2) \in X_{2v}, \\
i & \text{otherwise.}
\end{cases}
$$

Then $\{\kappa^x_1, \ldots, \kappa^x_c\}$ induces a complete subgraph of $K^{\Omega_{2w-1}(K_{2c-2})}$. Indeed for $i \neq j$, the intersection of the images of $\kappa^x_i$ and $\kappa^x_j$ is $\{1, 2\}$. If $(X_0, \ldots, X_{2v})$ and $(Y_0, \ldots, Y_{2v})$ are neighbours in $\Omega_{2w-1}(K_{2c-2})$, then $X_{2v}$ and $Y_{2v}$ are disjoint, hence they cannot contain the same element $(x, y)$. Moreover, for any $j \notin \{1, 2\}$, $\tau^x_{i, j}$ is adjacent to $\kappa^x_i$ for the same reason that $\tau^x_{i, j}$ is adjacent to $\tau^x_{k, i}$.

5.3. Subsets of $H_{\text{id}}$. For $x \in \{1, \ldots, c-1\}$, we now consider the subsets $N^0_x$, $N^1_x$, $\ldots$, $N^{2v}_x$, of elements of $H_{\text{id}}$ defined recursively as follows. $N^0_x = \Phi_0^{-1}(x)$, that is, the set of functions $f$ such that $f(\gamma) = x$. Then for $\ell \in \{1, \ldots, 2v\}$, $N^x_\ell$ is defined as the set of $f$ in $V(H_{\text{id}})$ a neighbour $g$ in $N^x_{\ell-1}$. The properties of the various elements $\gamma$, $\tau^x_{i, j}$, $\kappa^x_i$ of $K^{\Omega_{2w-1}(K_{2c-2})}$ discussed above have the following consequences.

(i) If $f \in N^0_x$, then $f(\gamma) = x$ by definition.

(ii) If $f \in N^x_\ell$ is adjacent to $g \in N^0_x$, then $f(\tau^x_{i, j}) \neq g(\gamma) = x$. Therefore $f(\tau^x_{i, j}) = c$.

(iii) Generally, if $f \in N^x_{\ell+1}$ is adjacent to $g \in N^x_\ell$, then $f(\tau^x_{i, j}) \neq g(\tau^x_{i, j})$.

Thus if $g(\tau^x_{i, j}) = i$, then $f(\tau^x_{i, j}) = k$.

(iv) Finally if $f \in N^x_{2v}$ is adjacent to $g \in N^x_{2v-1}$, then $f(\kappa^x_i) \neq g(\tau^x_{i, j})$. Thus if $g(\tau^x_{i, j}) = i$, then $f(\kappa^x_i) \in \{1, 2\}$.

The validity of the antecedent $g(\tau^x_{i, j}) = i$ in (iii) depends on the parameters $\ell, i, j$. Item (i) guarantees the validity of this antecedent only with $\ell = 1$, $i = c$ and $j = x$. But the free parameter $k$ allows to expand the set of values $i$, $j$ with $g(\tau^x_{i, j}) = i$ as $\ell$ increases: When $\ell = 2$, $i$ can be any value other than $x$, $c$, and $j$ must be $c$. When $\ell = 3$, $i$ can be any value other than $c$, and $j$ is any value other than $x, c$. When $\ell = 4$, the only restriction remaining is that $j$ cannot be $c$, and when $\ell \geq 5$, $i$ and $j$ can be any two distinct values in $\{1, \ldots, c\}$. Since $2v - 1 \geq 5$, the antecedent $g(\tau^x_{i, j}) = i$ is true in (iv), therefore $f(\kappa^x_i) \in \{1, 2\}$.

Note that $N^x_2$ contains $N^{2v}_{x-2}$ for $x \in \{2, \ldots, 2v\}$. We define $\Psi_x : N^x_v \to \{1, 2\}$ by letting $\Psi_x(f)$ be a colour repeated at least twice as $f(\kappa^x_i), i \in \{3, \ldots, c\}$. The map $\Phi_1 : H_{\text{id}} \to \{1, 2\}$ is then given by $\Phi_1(f) = \Psi_x(f)$ for $x = \Phi_0(f)$.

It remains to show that $\Phi = (\Phi_0, \Phi_1)$ is a $(v + 1)$-wide colouring of $H_{\text{id}}$. Let $f_0, \ldots, f_{2w+1}$ be a walk in $H_{\text{id}}$. If $\Phi_0(f_0) \neq \Phi_0(f_{2w+1})$, then $\Phi(f_0) \neq \Phi(f_{2v+1})$. Suppose that $\Phi_0(f_0) = \Phi_0(f_{2w+1}) = x$. Then $f_n \in N^x_v$ for all $n$. For $n \in \{0, \ldots, 2v\}$ there exist distinct values $i, j \in \{3, \ldots, c\}$ such that $\Psi_x(f_n) = f_n(\kappa^x_i) \neq f_n(\kappa^x_j) = \Psi_x(f_{n+1})$. Thus the values of $\Psi_x(f_n)$ alternates between 1 and 2 as $n$ goes from 0 to $2v + 1$. Therefore $\Phi_1(f_n) = \Psi_x(f_0) \neq \Psi_x(f_{2w+1}) = \Phi_1(f_{2v+1})$, and $\Phi(f_0) \neq \Phi(f_{2v+1})$. This concludes the proof.

5.4. Discussion. Even though we produce a $(v + 1)$-wide colouring of the subgraph $H_{\text{id}}$ of $K^{\Omega_{2v-1}(K_{2c-2})}$, the same method does not yield a $(2v + 1)$-wide colouring of the same graph. The proximity of the endpoints of the walk $f_0, \ldots, f_{2w+1}$ is
necessary to have the same $\Psi_x$ defined on the whole walk. Perhaps the arbitrary pairing of vertices of $K_{2c-2}$ that we use imposes this type of limitation on the conclusion. In any case, proving the hypothesis in Proposition 6 remains a desirable objective, even though the proof method may be quite different from our proof of Lemma 5.

However, it is also possible that the hypothesis and even the conclusion of Proposition 6 are false. In the next section, we explore the case of directed graphs and present one instance of an infinite exponential lattice that is atomic rather than dense.

6. An infinite exponential lattice that is atomic

For any graph $K$, $K^G$ is a Boolean lattice. Thus if $K^G$ is finite, its structure is uniquely determined by its cardinality. In particular, $K^G$ is a two-element lattice if and only if $K$ is multiplicative. In general, $K^G$ has $2^n$ elements if and only if there exist incomparable multiplicative graphs $M_1, \ldots, M_n$ such that $K \leftrightarrow \Pi_{i=1}^n M_i$ (see [19]). Using the fact that all square-free graphs are multiplicative (see [21]), we conclude that the exponential lattices $K^G$ can be isomorphic to any finite boolean lattice.

By Theorem 1, the exponential lattices $K^G_c, c \geq 5$ are countably infinite. Proposition 6 outlines sufficient conditions for these lattices to be dense. There is only one isomorphism class of dense boolean lattices, namely that of finite unions of intervals of the form $[x, y)$ with rational endpoints in $[0, 1)$. However a countably infinite Boolean lattice needs not be dense. In this section we explore the context of directed graphs, where countably infinite exponential lattices can be dense at one extreme or atomic at the other.

The concepts of homomorphisms, products and exponential graphs adapt readily to the context of directed graphs. Let $D$ be the class of finite directed graph, and $K^D$ the class of exponential digraphs $K^G, G \in D$, quotiented by $\leftrightarrow$. Note that $K_c^D$ contains a natural copy of $K_c^G$. In [18] it is shown that $K_c^D$ is dense for any $c \geq 3$. We present an example of a digraph $A$ such that $A^D$ is infinite but atomic.

The digraph $A$ has a simple presentation: its vertex-set is $\{1, 2, 3, 4\}$ and its arc-set is $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$. However it is useful to view $A$ as the “dual of a regular family of paths” in the sense of [4, 5]. A path $P$ with vertices labeled consecutively $v_0, \ldots, v_n$ corresponds to a word $w$ in $\{+,-\}^n$, where the $i$-th symbol is $+$ if $(v_{i-1}, v_i)$ is an arc of $P$, and $-$ if $(v_i, v_{i-1})$ is an arc of $P$. We then write $P = P(w)$. In this way, a language $L$ on the alphabet $\{+,-\}$ generates a family $\mathbb{P}(L)$ of paths.

Lemma 7 ([4], Example 3.3). A digraph $H$ admits a homomorphism to the digraph $A$ if and only if no path in $\mathbb{P}(A)$ admits a homomorphism to $H$, where

$$A = \{+++(+-+)^n + |n \in \mathbb{N}\}.$$

In the language of homomorphism duality, $A$ is thus the dual of $\mathbb{P}(A)$. Note that $A$ is a regular language, corresponding to the regular expression

$$\{++\} \circ \{-+\}^* \circ \{+\}.$$

More generally we have the following.
Lemma 8 ([5]). Any regular language $R \subseteq \{+, -\}^*$ admits a digraph dual, that is, a (finite) digraph $D(R)$ such that for any digraph $H$, $H$ admits a homomorphism $D(R)$ if and only if no path in $P(R)$ admits a homomorphism to $H$.

Also, for any finite digraph $H$, the set of words corresponding to paths which do not admit a homomorphism to $H$ is a regular language $R$, and $H$ admits a homomorphism to $D(R)$.

The paths in $P(A)$ form an antichain with respect to the relation $\rightarrow$. That is, there is no homomorphism between any two distinct paths in this family. This implies the following.

Proposition 9. The lattice $A_D$ is isomorphic to the lattice of regular languages contained in $A$, ordered by reverse inclusion.

Proof. Let $H$ be a digraph such that $H \not\rightarrow A$ and $A^H \not\rightarrow A$, so that both $A^H$ and $A^{A^H}$ are nontrivial elements of $A_D$. Then $P(A)$ contains at least one path that admits a homomorphism to $H$, and one that admits a homomorphism to $A^H$. No path $P$ in $P(A)$ admits a homomorphism to both $H$ and $A^H$, for otherwise we would have $P \rightarrow H \times A^H \rightarrow A$. Therefore, the language

$$R = \{ w \in A | P(w) \rightarrow A^H \}$$

is a nontrivial regular language properly contained in $A$.

We claim that $D(R) \leftrightarrow A^{A^H}$. Indeed by definition of $R$, no path in $P(A)$ admits a homomorphism to both $A^H$ and $D(R)$. Therefore $D(R) \times A^{A^H} \rightarrow A$ hence $D(R) \rightarrow A^{A^H}$. Also, since $A^H \times A^{A^H} \rightarrow A$, no path in $P(R)$ admits a homomorphism to $A^{A^H}$, hence $A^{A^H} \rightarrow D(R)$.

By the same argument (applied to $H' = A^H$ rather than $H$), $A^H \leftrightarrow D(R)$. Thus all elements of $A_D$ are of the form $D(L)$ for some regular language $L$ contained in $A$.

Thus $A_D$ is an atomic infinite Boolean lattice. Its coatoms are the “singleton duals” $D(\{P(w)\})$ and its atoms are $A^{D(\{P(w)\})} = D(\{P(w)\})$ with $w \in A$. It remains to see whether the lattices $K^D_c$, $c \geq 5$ are atomic like $A_D$, dense like $K^c_D$, $c \geq 3$, or have yet another structure.

References


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