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[GTN LIV:6] HEDETNIEMI'S CONJECTURE, 40 YEARS LATER

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Abstract

Hedetniemi's conjecture states that the chromatic number of a categorical product of graphs is equal to the minimum of the chromatic numbers of the factors. We survey the many partial results surrounding this conjecture, to review the evidence and the counter evidence.

1. Introduction

The *categorical product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where $(u, v)(u', v') \in E(G \times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. It is easy to derive a proper vertex coloring of $G \times H$ from a proper vertex coloring of G or of H. More than forty years ago, Hedetniemi conjectured that this is essentially the best way to color the categorical product of two graphs:

Conjecture 1.1 [1]: $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$,
where $\chi(K)$ is the *chromatic number* of the graph K. □

The year was 1966 and Hedetniemi was a graduate student at the University of Michigan. To put matters in perspective, the four color problem was still the biggest open problem in graph coloring, only probabilistic constructions were known for graphs with large girth and large chromatic number, and the concept of NP-completeness had not yet been formulated. Arguably, the supporting evidence for Hedetniemi's conjecture is scanty by today's standards. Yet the conjecture survived and the research that grew out of it over the years revealed the depth and richness of the subject of product colorings.

There have been two previous surveys **[2][3]** of Hedetniemi's conjecture. In the present survey, we classify partial results surrounding it as supporting evidence or counter evidence, and discuss stronger and weaker conjectures and related problems. For the sake of fluidity, we omit proofs and limit the presentation of some useful auxiliary concepts such as homomorphisms, topological bounds, fractional graph theory; the interested reader should consult the references provided on these subjects. Our goal is to give a clear picture of the modesty of our state of knowledge concerning colorings of products of general graphs, and indicate that the study of Hedetniemi's conjecture for specific classes of factors is perhaps worthy of more attention than it has received so far.

2. Reformulations

The inequality

$$\chi(G \times H) \le \min\{\chi(G), \chi(H)\}$$

follows easily from the fact that for any coloring c of a factor (say G), we can define a coloring c' of $G \times H$ by $c'(u, v) = c(u)$. The difficulty lies in deriving a coloring of a factor from a coloring of the product, to prove the second inequality,

$$\chi(G \times H) \ge \min\{\chi(G), \chi(H)\}.$$

It is convenient to consider each chromatic number separately, by introducing the statements

$\boldsymbol{H}(n)$: If $G \times H$ is n-colorable, then G or H is n-colorable.

Thus, Hedetniemi's conjecture holds if and only if $\boldsymbol{H}(n)$ holds for all n. This section of the paper consists of various reformulations of the conjecture in the language of graph homomorphisms.

2.1. Multiplicative Graphs

Following **[4]**, a *homomorphism* from G to K is a map $\phi\colon V(G) \to V(K)$ such that, if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(K)$. The statement: "there exists a homomorphism from G to K" is denoted by $G \to K$ and its negation by $G \nrightarrow K$. We write $G \leftrightarrow H$ when $G \to H$ and $H \to G$; G and H are then called *homomorphically equivalent*.

A graph K is called *multiplicative* if, whenever we have $G \times H \to K$, then $G \to K$ or $H \to K$. Let K_n denote the complete graph on n vertices, then $G \to K_n$ if and only if $\chi(G) \le n$. This allows the reformulation:

Proposition 2.1 [5]: $\boldsymbol{H}(n)$ is equivalent to the statement: K_n is multiplicative. ■

The adjective *multiplicative* has become standardized although the property was previously introduced under the name *productivity* **[6]**. Hedetniemi's conjecture falls within the more general problem of characterizing multiplicative graphs, digraphs, and relational structures (see also **[2][7]–[9]**). It is difficult to prove the multiplicativity of a single graph, since this property depends on the structure of the whole category of graphs. In fact, the question as to whether a graph is multiplicative is not even known to be decidable. *Exponential graphs*, introduced next, have been the most successful tools in the study of multiplicative graphs.

2.2. Exponential Graphs

Given two graphs G and K, the *exponential graph* K^G has for vertices the set of all functions $\varphi\colon V(G) \to V(K)$ (not just homomorphisms), and edges the pairs (φ_1, φ_2) such that for every $uv \in E(G)$, $\varphi_1(u)\varphi_2(u) \in E(K)$. Exponentiation and exponential structures have many applications in algebra and elsewhere. It follows from the definition of K^G that the *evaluation map* $\varepsilon\colon G \times K^G \to K$ defined by $\varepsilon(u, \varphi) = \varphi(u)$ is a homomorphism. Moreover, we have $G \times H \to K$ if and only if $H \to K^G$. Thus, for an integer n and a graph G, there exists a graph H such that $\chi(G \times H) \le n$ if and only if $H = K_n^G$ has this property. This implies:

Proposition 2.2 [10]: $\boldsymbol{H}(n)$ is equivalent to the statement:
If $\chi(G) > n$, then $\chi(K_n^G) = n$. ■

In **[5][10][11]** this version is exploited, not by considering K_n^G itself, but suitable subgraphs of K_n^C, where $C \to G$. Indeed, if $C \to G$, then $K_n^G \to K_n^C$ and thus it is sufficient to color a subgraph of K_n^C containing a homomorphic image of K_n^G. It is not clear that this method can be generalized to prove more cases of Hedetniemi's conjecture. However, a second enlightening use of exponentiation is to apply it to an entire category of graphs rather than to a single graph.

Let $K_n^{\mathcal{G}}$ be the set of all graphs of the form K_n^G. The relation $\to$ induces a preorder on $K_n^{\mathcal{G}}$. The quotient structure $K_n^{\mathcal{G}}/\!\leftrightarrow$ is well known to be a Boolean lattice (see **[8]**). Thus, in a sense, the structure $K_n^{\mathcal{G}}/\!\leftrightarrow$ is much better understood than that of the single elements in $K_n^{\mathcal{G}}$. $\boldsymbol{H}(n)$ is equivalent to the statement that $K_n^{\mathcal{G}}/\!\leftrightarrow$ is a two-element lattice. The maximal element of $K_n^{\mathcal{G}}/\!\leftrightarrow$ consists of all graphs K_n^G containing loops; that is, all graphs K_n^G such that $G \to K_n$. (The homomorphisms $\varphi\colon G \to K_n$ are the loops of K_n^G.) The minimal element of $K_n^{\mathcal{G}}/\!\leftrightarrow$ consists of all graphs K_n^G such that $K_n^G \to K_n$; conjecturally, this coincides with the class of all graphs K_n^G such that $G \nrightarrow K_n$.

2.3. Retracts and Products

A graph K is called a *retract* of a graph G if there exist homomorphisms $\rho\colon G \to K$ and $\gamma\colon K \to G$ such that $\rho \circ \gamma$ is the identity on K. In particular, if the complete graph K_n is a retract of G, then $\chi(G) = \omega(G) = n$, where $\omega(G)$ is the *clique number* of G. For general graphs, the clique number is often smaller than the chromatic number. Thus, to express the chromatic number of a graph G in terms of retracts we need to add a disjoint copy of K_n: $\chi(G) \le n$ if and only if K_n is a retract of the disjoint union of G and K_n. Doing this to both factors of a product, we obtain the following.

Proposition 2.3 [12]: $\boldsymbol{H}(n)$ is equivalent to the statement: Whenever K_n is a retract of a product of graphs, it is a retract of one of the factors. ■

A graph is called a *core* if it has no proper retracts. Every (finite) graph has a core that is unique up to isomorphism, and Proposition 2.3 generalizes to a characterization of the multiplicative cores as the cores that cannot be expressed as a retract of a product without being a retract of a factor. In many structure theories, one objective is to similarly characterize "irreducible" elements that cannot be built up as retracts of products in a nontrivial way.

The statement can also be presented from an order-theoretic point of view: The relation $\to$ induces a preorder on the category $\mathcal{G}$ of all graphs. It is well known (see **[4]**) that the natural quotient $\mathcal{G}/\leftrightarrow$ is a distributive lattice, with the meet operation induced by the categorical product and the join operation induced by the disjoint union. The multiplicative graphs turn out to correspond to the meet-irreducible elements of $\mathcal{G}/\leftrightarrow$. Thus Hedetniemi's conjecture states that the complete graphs are meet-irreducible.

3. Supporting Evidence

The statements $\boldsymbol{H}(1)$ and $\boldsymbol{H}(2)$ are relatively trivial. $\boldsymbol{H}(3)$ is nontrivial and was proved by El-Zahar and Sauer.

> **Theorem 3.1 [10]:** The chromatic number of the product of two 4-chromatic graphs is 4. ■

Note that the formulation of Theorem 3.1 (which is the title of **[10]**) is contrapositive. The exposition in **[10]** is closer to a direct proof of $\boldsymbol{H}(3)$, through a 3-coloring of some connected components of K_3^C, where C is an odd cycle contained in G or H. In **[13]**, it is shown that this gives a polynomial procedure to derive a 3-colouring of G or of H from a 3-coloring of $G \times H$.

None of the other statements $\boldsymbol{H}(n)$ are proved. However, many partial results for large chromatic numbers have been obtained by restricting the class of factors considered.

> **Theorem 3.2 [14]:** Let G be a graph such that every vertex of G is in an n-clique. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

> **Theorem 3.3 [15]:** Let G be a graph such that $\chi(G) \geq n$ and for every pair e_1, e_2 of edges of G, there is an edge e_3 incident to both of them. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

The *Hajos sum* of two graphs G and H with respect to the edges $[uv \in E(G)$ and $u'v' \in E(H)$ is the graph obtained from the disjoint union of G and H by removing uv and $u'v'$, identifying u and u' to a single vertex, and adding the edge vv'.

> **Theorem 3.4 [16]:** Let G be a Hajos sum of two graphs A and B, where $\chi(K_n^A) = n$ and B is obtained from copies of K_{n+1} by means of adding vertices and edges, taking Hajos sums, and at most one identification of nonadjacent vertices. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

In the preceeding three results, the factor G is heavily constrained (to find an n-coloring of K_n^G, but the factor H is free. The next results show that it is possible to impose weaker conditions, but on both factors at the same time.

> **Theorem 3.5 [17][18]:** Let G and H be connected graphs containing n-cliques. If $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

> **Theorem 3.6 [11]:** Let G and H be connected graphs containing odd wheels. If $\chi(G \times H) = 4$, then $\min\{\chi(G), \chi(H)\} = 4$. ■

It is interesting to compare Theorem 3.5 with the products and retracts version of Hedetniemi's conjecture. Theorem 3.5 can be shown to be equivalent to the statement that whenever the complete graph K_n is a retract of a product of two connected graphs, it is a retract of a factor. By Proposition 2.3, if we drop the requirement that the factors be connected, we get a statement that is equivalent to Hedetniemi's conjecture. Theorem 3.1 can be seen as a stronger version of Theorem 3.5, when $n = 3$, replacing the condition that the factors contain triangles by the (trivial) condition that the factors contain odd cycles. This point of view inspired Theorem 3.6 from Theorem 3.5 when $n = 4$, replacing the condition that the factors contain 4-cliques by the condition that the factors contain odd wheels. However, this approach has serious limitations (see Theorem 4.3).

Other results of this type involve well known lower bounds for the chromatic number. In **[19]**, the value of two plus "the connectivity of the geometric realization of the neighborhood complex of a graph" is introduced as a lower bound on the chromatic number of a graph. Here we call it the *topological bound*, although there are many similarly defined "topological bounds" (see **[20]**).

Theorem 3.7 [21]: Let G and H be graphs for which the topological bound on the chromatic number is tight. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. ■

Another such bound is the *fractional chromatic number* $\chi_f(G)$ of a graph G; that is, the common linear relaxation of its clique number and its chromatic number (under suitable integer programming formulations, see **[22]** and Section 7). In particular, for every graph G, we have $\chi_f(G) \leq \chi(G)$.

Theorem 3.8 [23]: $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/2$. ■

The partial results of this section are susceptible of refinements and variations. It is unlikely that any of them will yield a complete proof of Hedetniemi's conjecture, but even partial results are interesting in their own right. For instance, suppose $\chi(G) = \chi(H) = n$, the topological bound is tight on G and $\chi_f(H) = n$, then Hedetniemi's conjecture states that $\chi(G \times H) = n$ but a common refinement of Theorems 3.7 and 3.8 would be needed to confirm this. This suggests the following.

Problem 3.9: Is there a natural lower bound on the chromatic number of a graph that is a common refinement of the topological bound and the fractional chromatic number? □

In the same vein, consider the following two results.

Theorem 3.10 [24]: Let G be a *Cayley graph* on $\mathbb{Z}_2^n$. If G contains an odd cycle, then $\chi(G) \geq 4$. ■

Theorem 3.11 [25]: Let G be a *vertex-transitive* graph such that G contains a triangle and $|V(G)|$ is not a multiple of 3. Then $\chi(G) \geq 4$. ■

If G and H satisfy the hypotheses of Theorems 3.10 and 3.11, respectively, then by Theorem 3.1, $\chi(G \times H) \geq 4$. It would be interesting to prove this result directly in the context of vertex-transitive graphs.

Problem 3.12: Is there a natural class of non 3-colorable vertex-transitive graphs that generalizes both the hypotheses of Theorem 3.10 and Theorem 3.11? □

4. Stronger Conjectures

4.1. Fiber Products

El-Zahar and Sauer actually proved a stronger result than Theorem 3.1.

Theorem 4.1 [10]: Let G and H be connected 4-chromatic graphs, and let C and C' be odd cycles contained in G and H, respectively. Then the subgraph of $G \times H$ induced by $(C \times H) \cup (G \times C')$ is 4-chromatic. ■

They conjectured that a similar phenomenon holds for higher chromatic numbers.

Conjecture 4.2 [10]: Let G and H be connected $(n+1)$-chromatic graphs, and let G' and H' be n-chromatic subgraphs of G and H, respectively. Then the subgraph of $G \times H$ induced by $(G' \times H) \cup (G \times H')$ is $(n+1)$-chromatic. □

This, however, turns out to be false.

Theorem 4.3 [11]: There exists a 4-chromatic graph K such that for each $n \geq 5$ there exists an n-chromatic graph G_n containing K as a subgraph, such that the subgraph of $G_n \times G_n$ induced by $(K \times G_n) \cup (G_n \times K)$ is 4-chromatic. ■

The fiber product yields another hypothesis on the structure of n-chromatic subgraphs of products of n-chromatic graphs. Given two n-chromatic graphs with n-colorings $c_G: G \to K_n$ and $c_H: H \to K_n$, their *fiber product* over c_G and c_H is the subgraph $(G, c_G) \times (H, c_H)$ of $G \times H$ induced by

$$V((G, c_G) \times (H, c_H)) = \{(u, v) \in V(G \times H) : c_G(u) = c_H(v)\}.$$

Conjecture 4.4 [26]: The fiber product of two n-chromatic graphs over n-colorings is n-chromatic. □

In **[26]**, the conjecture is proved for $n = 3$; it is still open for $n = 4$. Note that both Conjectures 4.2 and 4.4 suggest that the critical subgraphs of $G \times H$ are smaller that $G \times H$ itself.

4.2. Uniquely Colorable Graphs

In **[27]** it is proved that if G is connected and $\chi(G) > n$, then $G \times K_n$ is uniquely n-colorable. This motivates the following refinements of $\boldsymbol{H}(n)$ in terms of *uniquely colorable graphs*.

> $\boldsymbol{A}(n)$: If G and H are uniquely n-colorable, then each n-coloring of $G \times H$ is induced by a coloring of G or H.
>
> $\boldsymbol{B}(n)$: If G is uniquely n-colorable, H is connected, and $\chi(H) > n$, then $G \times H$ is uniquely n-colorable.

In **[17]** it is proved that $\boldsymbol{A}(n)$ implies $\boldsymbol{B}(n)$ and $\boldsymbol{B}(n)$ implies $\boldsymbol{H}(n)$. It is also conjectured that $\boldsymbol{A}(n)$ holds for all n.

4.3. Circular Colorings

Following **[28]**, for relatively prime integers r, s such that $2r \le s$ the *circular complete graph* $K_{s/r}$ has the elements of $\mathbb{Z}_s$ for vertices and for edges the pairs (i, j) such that $j - i \in \{r, r+1, \ldots, s-r\}$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the (well-defined) smallest rational q such that $G \to K_q$. In particular, $\chi(G) = \lceil \chi_c(G) \rceil$; hence, χ_c is a refinement of χ. Zhu proposed the following strengthening of Hedetniemi's conjecture:

> **Conjecture 4.5 [28]:** $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$. □

The result of El-Zahar and Sauer has been adapted to the circular case, as well as the result of Duffus–Sands–Woodrow and Welzl.

> **Theorem 4.6 [9]:** If $\min\{\chi_c(G), \chi_c(H)\} \le 4$, then $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$. ■
>
> **Theorem 4.7 [29]:** If G and H are connected graphs containing $K_{s/r}$ and $\chi_c(G \times H) = s/r$, then $\min\{\chi_c(G), \chi_c(H)\} = s/r$. ■

To date, the only graphs known to be multiplicative (up to homomorphic equivalence) are the circular complete graphs K_1 and K_q, $2 \le q < 4$.

5. Counter Evidence

There is no known counterexample to Hedetniemi's conjecture. However, there are natural extensions of this conjecture that are known to be false, most notably for directed graphs and infinite chromatic numbers.

5.1. Infinite Graphs

The chromatic number of an infinite graph is a (finite or infinite) cardinal. The product of two infinite chromatic graphs is still infinite chromatic, but nonetheless Hajnal notes that Hedetniemi's conjecture fails for infinite chromatic numbers.

> **Theorem 5.1 [30]:** For every infinite cardinal κ, there exist graphs G and H such that $\chi(G) = \chi(H) = \kappa^+$ and $\chi(G \times H) = \kappa$. ■

The examples provide an interesting application of the theory of stationary sets, although they cannot be directly adapted to the finite case. Extensions of Theorem 5.1 involve models of set theory; Soukup **[31]** proved that the following statement:

> There are two graphs G and H both of cardinality $\aleph_2$ such that $\chi(G) = \chi(H) = \aleph_2$ and $\chi(G \times H) = \aleph_0$.

is consistent with the axioms of Zermelo–Fraenkel set theory, with the axiom of choice, and the generalized continuum hypothesis, although Hajnal notes that it cannot be proved in this axiom system.

In the finite case, Hedetniemi's conjecture implies that the formula

$$\chi\left(\prod_{i=1}^{n} G_i\right) = \min\{\chi(G_1), \ldots, \chi(G_n)\}$$

is valid for any finite n. Miller **[32]** notes that it cannot be extended to an infinite number of factors. Indeed, the product of any family of graphs with unbounded odd girth is bipartite.

5.2. Directed Graphs

A coloring c of the vertices of a directed graph $\vec{G}$ is called *proper* if for every arc $\vec{(u, v)}$, $c(u) \neq c(v)$. Thus the constraints are not affected by the orientation, and the minimum number $\chi(\vec{G})$ of colors needed to properly color $\vec{G}$ is just the chromatic number of the underlying undirected graph obtained from $\vec{G}$. The *categorical product* $\vec{G} \times \vec{H}$ of two directed graphs $\vec{G}$ and $\vec{H}$ has vertex set $V(\vec{G}) \times V(\vec{H})$, and its arcs are the couples $((u, u'), (v, v'))$ such that (u, v) is an arc of $\vec{G}$ and (u', v') is an arc of $\vec{H}$. Categorical products of directed graphs have fewer edges than their undirected counterparts and tend to be easier to color. Poljak and Rödl **[33]** noted that Hedetniemi's conjecture fails for directed graphs, even with relatively small examples. To date, the strongest results in this direction are the following.

> **Theorem 5.2 [34]:** For every $\varepsilon > 0$ and $N > 0$, there exists an integer $n > N$ and directed graphs $\vec{G}_n$ and $\vec{H}_n$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = n$ and $\chi(\vec{G}_n \times \vec{H}_n) \leq \frac{2}{3}n + \varepsilon$. ■

> **Theorem 5.3 [35][36]:** For each $n \geq 4$, there exist directed graphs $\vec{G}$ and $\vec{H}$ such that $\chi(\vec{G}_n) = n$, $\chi(\vec{H}_n) = 4$, and $\chi(\vec{G}_n \times \vec{H}_n) = 3$. ■

> **Theorem 5.4 [35]:** There exist directed graphs $\vec{G}$ and $\vec{H}$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = 5$ and $\chi(\vec{G}_n \times \vec{H}_n) = 3$. ■

The *Poljak–Rödl function* $f: \mathbb{N} \to \mathbb{N}$ is defined by

$$f(n) = \min\{\chi(\vec{G} \times \vec{H}) : \chi(\vec{G}) = \chi(\vec{H}) = n\}.$$

Its undirected counterpart is the function $g: \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}.$$

Thus, Hedetniemi's conjecture states that $g(n) = n$ for all n, whereas Theorem 5.2 shows that asymptotically $f(n) \leq 2n/3$. The two functions are obviously nondecreasing, and $f(n) \leq g(n)$ for all n. The values $f(1) = g(1) = 1$, $f(2) = g(2) = 2$, $f(3) = g(3) = 3 = f(4) = f(5)$, and $g(4) = 4$ are the only known values of f and g. In fact, the functions are not even known to be unbounded. Building on Poljak and Rödl's work with the *arc-graph construction*, Poljak, Schmerl,[1] and Zhu independently proved the following intriguing statements.

> **Proposition 5.5 [3][37]:**
> (1) Either f is unbounded or $f(n) \leq 3$ for all n.
> (2) Either g is unbounded or $g(n) \leq 9$ for all n. ■

Theorem 5.3 suggests that f may be bounded. However, this would contradict Hedetniemi's conjecture.

> **Proposition 5.6 [38]:** f is bounded if and only if g is bounded. ■

Perhaps the strongest argument against Hedetniemi's conjecture is the difficulty in proving that g is unbounded. Many interesting developments center around this question (see Section 6).

5.3. Hypergraphs

Following **[39]**, the *box product* of two graphs G and H is the hypergraph $G■H$ with vertex set $V(G) \times V(H)$ and hyperedges $\{uu', uv', vu', vv'\}$ for every $uv \in E(G)$ and $u'v' \in E(H)$. A coloring of the vertices of a hypergraph is called *proper* if it has no monochromatic hyperedge, and the chromatic number of a hypergraph is the minimum number of colors needed to properly color the hypergraph.

Mubayi and Rödl **[39]** propose a general conjecture that admits the following particular case.■

> **Conjecture 5.7:** There exists a bound c such that for every n, there exist graphs G_n and H_n such that $\chi(G_n) = \chi(H_n) = n$ and $\chi(G_n■H_n) \leq c$. □

Note that a proper coloring of $G■H$ can be derived from a proper coloring of any product $\vec{G}_n \times \vec{H}_n$ of an orientation of G and orientation of H. Thus, the *fallacy* of Conjecture 5.7 would imply that the Poljak-Rödl function f is unbounded. Nonetheless Mubayi and Rödl base their conjecture on the fact that the statement is true for products of hypergraphs, when the size of the hyperedges of the factors is not bounded.

[1] J. Schmerl's unpublished result was obtained at the 1984 Banff conference *Graphs and Order*, after hearing of Poljak and Rödl's result. His contribution is mistakenly attributed to Schelp in **[3]**.

Theorem 5.8 [39]: For every integer n there exists a hypergraph H_n such that $\chi(H_n) \geq n$ and $\chi(H_n \blacksquare H_n) = 2$. ■

This refutes a conjecture of **[40]**, although **[41]** shows that variants of Theorems 3.2 and 3.5 can be adapted to products of hypergraphs. In a sense, this weakens the support that Theorems 3.2 and 3.5 offer to Hedetniemi's conjecture.

5.4. Multicolorings

For an integer r, an r-coloring of a graph G is an assignment of an r-set $c(u)$ to each vertex u of G, such that if $uv \in E(G)$, then $c(u) \cap c(v) = \varnothing$. The *r-chromatic number* $\chi_r(G)$ of G is the least integer s such that G admits an r-coloring c with $\left|\bigcup_{u \in V(G)} c(u)\right| = s$. In particular, a 1-coloring is an ordinary coloring and $\chi_1(G) = \chi(G)$.

An r-coloring of $G \times H$ can be derived from an r-coloring of G or H, but also from the disjoint union of an r_1-coloring of G with an r_2-coloring of H, where $r_1 + r_2 = r$. This is the basis of the following result:

Theorem 5.9 [42]: There are graphs G and H and integers r such that $\chi_r(G \times H) < \min\{\chi_r(G), \chi_r(H)\}$. ■

Multichromatic numbers can also be defined in terms of homomorphisms to Kneser graphs. For integers r and s, the *Kneser graph* $K(r, s)$ has the r-subsets of $\{1, 2, \ldots, s\}$ as vertices and two vertices are adjacent if they correspond to disjoint subsets. Thus, for a graph G, $\chi_r(G)$ is the least integer s such that G admits a homomorphism in $K(r, s)$. Hence, Theorem 5.9 admits the following reformulation.

Theorem 5.10 [42]: There are nonmultiplicative Kneser graphs. ■

Nonetheless, the adaptation of Theorem 3.5 is valid for multichromatic numbers.

Theorem 5.11 [12]: Whenever a Kneser graph is a retract of a product of connected graphs, it is a retract of a factor. ■

Theorems 5.10 and 5.11 weaken the support that Theorem 3.5 gives to Hedetniemi's conjecture.

6. Weaker Conjectures

6.1. Ramsey Theory

Burr, Erdős, and Lovász **[14]** rediscovered Hedetniemi's conjecture independently, and applied it to a problem in Ramsey Theory. For any graph G, there exists at least one graph F such that whenever the edges of F are colored in red and blue, then F contains a copy of G with all its edges having the same color. The smallest possible chromatic number for such a graph F is called the *chromatic Ramsey number* $r_c(G)$ of G. These authors proved the inequality $r_c(G) \geq (\chi(G) - 1)^2 + 1$, and proposed the following conjecture.

Conjecture 6.1 [14]: For every integer $r \geq 1$, there exists a graph G_r that $\chi(G_r) = r$ and $r_c(G_r) = (r-1)^2 + 1$. □

There is a natural candidate G_r constructed as follows. For every coloring p_i of the edges of $K_{(r-1)^2+1}$ in red and blue, there exists a monochromatic subgraph H_i of $K_{(r-1)^2+1}$ such that $\chi(H_i) \geq r$. Let G_r be the categorical product of all the graphs H_i. Assuming Hedetniemi's conjecture, then $\chi(G_r) = r$, and using this hypothesis, Burr, Erdős and Lovász **[14]** conclude that G_r verifies Conjecture 6.1. Theorem 3.2 is used to prove Conjecture 6.1 for $r \leq 4$. In **[43]**, this is extended to the case $r = 5$.

6.2. The Weak Hedetniemi Conjecture

The main open problem surrounding Hedetniemi's conjecture is the asymptotic behavior of the Poljak–Rödl functions f and g.

Conjecture 6.2 (The Weak Hedetniemi Conjecture): For every integer n, there exists an integer m_n such that if $\chi(G) = \chi(H) = m_n$, then $\chi(G \times H) \geq n$. □

By Theorem 5.5, it suffices to prove the result for $n = 10$, or a directed version of the result for $n = 4$. Thus, a strengthening of Theorem 3.1 to prove $\boldsymbol{H}(9)$ would be sufficient to prove Conjecture 6.2. Duffus and Sauer **[44]** observe that a common strengthening of Theorems 3.2 and 3.5 would also be sufficient.

Proposition 6.3 [44]: Consider the following hypothesis:
For any integer n and for any graphs G and H such that G is connected, G contains K_n, and $\min\{\chi(G), \chi(H)\} > n$, then $\chi(G \times H) > n$.
If this hypothesis is true, then the weak Hedetniemi conjecture is also true. ■

Let $\mathcal{G}$ be the class of finite undirected graphs. The lattice $K_9^{\mathcal{G}}/\leftrightarrow$ is well known to be Boolean (see **[8]**). If the lattice is finite or even if it has an atom $G/\leftrightarrow$ then for $n > \chi(K_9^{\mathcal{G}})$, $g(n) \geq 10$, whence, by Proposition 5.5, g is unbounded and the weak Hedetniemi conjecture is true. If, however, $K_9^{\mathcal{G}}/\leftrightarrow$ has no atoms, then it is dense, and up to isomorphism there is only one countable dense Boolean lattice. Therefore, at least one of the following statements holds.

(1) The weak Hedetniemi conjecture is true.

(2) The lattice $K_9^{\mathcal{G}}/\leftrightarrow$ is the unique dense countable Boolean lattice.

Both eventualities suggest a rich, and as yet undiscovered, structure in the category of graphs. Now, let $\mathcal{D}$ be the class of finite directed graphs. By Theorem 5.3, the lattice $K_3^{\mathcal{D}}/\leftrightarrow$ is at least known to be infinite. Perhaps it is possible to determine its exact structure.

Problem 6.4: Is $K_3^{\mathcal{D}}/\leftrightarrow$ isomorphic with the unique dense countable Boolean lattice? □

The weak Hedetniemi conjecture has interesting potential within the field of graph coloring (see Problem 3.9), but also in other fields of mathematics. In particular, Schmerl **[45]** linked the weak Hedetniemi conjecture to models of *Peano arithmetic*.

Proposition 6.5 [45]: Let $\mathcal{M}$ be an arithmetically saturated model of Peano arithmetic that is not a model of true arithmetic. Then $\mathcal{M}$ has a generic automorphism if and only if the weak Hedetniemi conjecture holds. ■

The weak Hedetniemi conjecture is reasonable, but so is the hypothesis that the Poljak–Rödl function g is bounded by 4. Indeed, El-Zahar and Sauer's proof of $\boldsymbol{H}(3)$ relies on odd cycles, hence on the fact that the topological bound is (essentially) tight for 3-chromatic graphs, whereas it is not always tight for larger chromatic numbers. However, it is harder to imagine the true bound on the Poljak–Rödl function being a number larger than 4. The case of directed graphs does not present such unacceptable alternatives: Either the function f is unbounded, or $f(n) = 3$ for all $n \geq 3$. Thus, it would be interesting to refine Proposition 5.5, and eventually prove that g is either unbounded or bounded by 4. The circular versions of the Poljak–Rödl functions considered next provide insight in this direction.

6.3. The Circular Weak Hedetniemi Conjecture

The circular versions of the Poljak–Rödl functions are the functions $f_c, g_c: [2, \infty) \to [2, \infty)$ defined by

$$f_c(x) = \inf\{\chi_c(\vec{G} \times \vec{H}) : \chi_c(\vec{G}), \chi_c(\vec{H}) \geq x\},$$

$$g_c(x) = \inf\{\chi_c(G \times H) : \chi_c(G), \chi_c(H) \geq x\}.$$

Obviously, f_c and g_c are bounded if and only if f and g are. By Theorem 4.6, we have $g(x) = x$ for $x \in [2, 4]$. Thus, if g_c is bounded, the bound is somewhere between 4 and 9. If f_c is bounded, the bound is at most 3, but to date there is not even a proof that f_c is not identically equal to 2. We can formulate the conjecture that f_c is not identically equal to 2 in terms of homomorphisms to odd cycles as follows.

Conjecture 6.6 (The Circular Weak Hedetniemi Conjecture): There exists an odd cycle C_{2k+1} and an integer n such that if $\chi_c(\vec{G}), \chi_c(\vec{H}) \geq n$, then there is no homomorphism from $\vec{G} \times \vec{H}$ to C_{2k+1}. □

Hedetniemi's conjecture trivially implies the weak Hedetniemi conjecture, which implies that f and f_c are unbounded (by Proposition 5.6), which implies Conjecture 6.6. However, even the latter seems to be a hard problem.

Tighter bounds on f_c would also improve the bounds on g, as shown by the following result.

Proposition 6.7: If f_c is bounded by 5/2, then g is bounded by 8, and if f_c is bounded by 7/3, then g is bounded by 7.

Proof: We include a sketch of proof here, since this result has not appeared elsewhere. Suppose that f_c is bounded by 5/2. Then, for every $n \geq 3$, there exist directed graphs $\vec{G}_n$ and $\vec{H}_n$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = n$ and there exists a homomorphism $\phi_n: \vec{G}_n \times \vec{H}_n \to C_5$. Put

$$m_n = \max\{\chi(K_{n-1}^{\vec{G}_n}), \chi(K_{n-1}^{\vec{H}_n})\} + 1 .$$

The undirected graphs A_n and B_n are defined as follows: A_n is obtained from $\vec{G}_n \times \vec{G}_{m_n}$ by ignoring its orientation, and B_n is obtained by ignoring the orientation in $\vec{H}_n \times \vec{R}$, where $\vec{R}$ is the digraph obtained from $\vec{H}_{m_n}$ by reversing its orientation. By definition of m_n, we have $\chi(A_n) = \chi(B_n) = n$. Define a function $\psi_n: V(A_n \times B_n) \to V(C_5 \times C_5)$ by

$$\psi_n((u, v), (w, x)) = (\phi_n(u, w), \phi_{m_n}(v, x)) .$$

Let $((u, v), (w, x)), ((u', v'), (w', x'))$ be an edge of $A \times B$. Without loss of generality we can assume that $((u, v), (u', v'))$ is an arc of $\vec{G}_n \times \vec{G}_{m_n}$. If $((w, x), (w', x'))$ is also an arc of $\vec{H}_n \times \vec{R}$, then $((u, w), (u', w'))$ is an arc of $\vec{G}_n \times \vec{H}_n$, whence $\phi_n(u, w)\phi_n(u', w')$ is an edge of C_5. Otherwise, $((w', x'), (w, x))$ is an arc of $\vec{H}_n \times \vec{R}$, whence $((v, x), (v', x'))$ is an arc of $\vec{G}_{m_n} \times \vec{H}_{m_n}$, and $\phi_{m_n}(v, x)\phi_{m_n}(v', x')$ is an edge of C_5. Therefore, n is a homomorphism from $A_n \times B_n$ to the *very strong product* $C_5 \star C_5$ defined by

$$E(C_5 \star C_5) = \{((i, j), (i', j')) : ii' \in E(C_5) \text{ or } jj' \in E(C_5)\}.$$

Therefore, $\chi(A_n \times B_n) \leq \chi(C_5 \star C_5)$. It is known (see **[46]**) that $\chi(C_5 \star C_5) = 8$ and this proves the first statement. The second statement is proved in a similar way, using the fact that $\chi(C_7 \star C_7) = 7$. ■

For every k we have $\chi(C_{2k+1} \star C_{2k+1}) \geq 7$. Therefore, the proof method of Proposition 6.7 cannot exhibit further links between the potential upper bounds of f_c and g.

7. Related Problems

7.1. Multiplicative Graphs

Many variations and strengthenings of Hedetniemi's conjecture, such as Conjecture 4.5, fall within the larger framework of characterizing multiplicative graphs in general. The problem is indeed interesting and deep, in view of the scarcity of known examples. Delhommé and Sauer **[7]** have shown that the class of square-free graphs is a promising candidate, where it might be possible to adapt the methods of **[10]**. In particular, they prove that if G and H are connected graphs each containing a triangle and K is square-free, then $G \times H \to K$ implies $G \to K$ or $H \to K$. This suggests the following question.

Problem 7.1: Are all square-free graphs multiplicative? □

Of course, the multiplicativity of square-free graphs does not impact directly on Hedetniemi's conjecture, but clearly a better understanding of the class of multiplicative graphs would be an asset, and the work in **[9]** shows that it is sometimes possible to prove the multiplicativity of some graphs by using the known multiplicativity of other graphs.

Other variations on Hedetniemi's conjecture are independent of multiplicativity. We present two of them next.

7.2. The Fractional Chromatic Number

The fractional chromatic number of a graph can also be defined in terms of multicolorings or homomorphisms into Kneser graphs:

$$\chi_f(G) = \min_{r \in \mathbb{N}} \frac{\chi_r(G)}{r} = \min\left\{\frac{s}{r} : G \to K(s, r)\right\}.$$

The fractional version of Hedetniemi's conjecture is the following.

Problem 7.2: Does the identity $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ always hold? □

Note that the identity would follow directly from the multiplicativity of all Kneser graphs; however, by Theorem 5.10, there are nonmultiplicative Kneser graphs. Nonetheless, in **[47]**, it is shown that the identity $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ holds whenever one factor belongs to a well behaved class of graphs, and in **[36]**, it is shown that the inequality $\chi_f(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/4$ always holds. Thus, there are some grounds for believing that the identity is always true.

When G is vertex-transitive, then $\chi_f(G) = |V(G)|/\alpha(G)$, thus Problem 7.2 admits as a subquestion the vertex-transitive case of the problem of determining of the independence number in categorical products of graphs. This is a very active research area **[29][48]–[50]**, where not only the cardinality of large independent sets but also their structure is analyzed.

The examples in Theorem 5.2 are tournaments, which implies that the directed version of the fractional Hedetniemi conjecture fails. The fractional version of the Poljak–Rödl functions

$$f_f(x) = \inf\{\chi_f(\vec{G} \times \vec{H}) : \chi_f(\vec{G}), \chi_f(\vec{H}) \geq x\},$$

$$g_f(x) = \inf\{\chi_f(G \times H) : \chi_f(G), \chi_f(H) \geq x\},$$

are essentially linear. Using lexicographic products with complete graphs, it can be shown (see **[36]**) that $g_f(x) \simeq c \cdot x$, with $c \in [1/4, 1]$, and $f_f(x) \simeq c \cdot x$ with $c \in [1/4, 2/3]$.

Either Hedetniemi's conjecture or its fractional version would imply the inequality $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}$. The inequality $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/2$ of Theorem 3.8 is known to hold both for directed and undirected graphs. The interesting phenomenon arising in these fractional versions of Hedetniemi's conjecture is that the lower bounds obtained so far for undirected graphs also hold for directed graphs. Distinguishing the undirected case from the directed case could perhaps help to understand the structure of independent sets in exponential graphs, and possibly impact on the non-fractional Hedetniemi conjecture.

7.3. The Local Chromatic Number

The *local chromatic number* of a graph G is the value

$$\chi_l(G) = \min\{\max_{u \in V(G)} |\{c(v) : uv \in E(G)\}| + 1 : c \text{ is a proper coloring of } G\},$$

that is, the maximum number of colors used "locally" (around a vertex) in a proper coloring of G. It can also be presented in terms of homomorphisms, by defining the graph $L(r, s)$ of "local r-colorings with s colors" as follows. The vertices of $L(r, s)$ are the couples (i, A) where A is an r-subset of $\{1, \ldots, s\}$ and $i \in A$, and the edges of $L(r, s)$ are the pairs $((i, A), (j, B))$ such that $i \in B$ and $j \in A$. Thus,

$$\chi_l(G) = \min\{r : \text{there exists } s \text{ such that } G \to L(r, s)\}.$$

At the 2007 *Canadian Discrete and Algorithmic Mathematics* conference in Banff, G. Simonyi asked about the validity of the local version of Hedetniemi's conjecture:

Problem 7.3: Does the identity $\chi_l(G \times H) = \min\{\chi_l(G), \chi_l(H)\}$ always hold? □

As in the fractional version, the identity in Problem 7.3 would follow from the multiplicativity of the graphs $L(r, s)$, although it does not depend on it. In **[51]** the inequalities $\chi_f(G) \leq \chi_l(G) \leq \chi(G)$ are proved, and in **[52]** another lower bound is obtained; namely, one half of (one version of) the topological bound on the chromatic number. Therefore, $\chi_l(G \times H)$ could differ significantly from $\min\{\chi_l(G), \chi_l(H)\}$ only both the fractional chromatic number and the topological bound are poor estimates for the chromatic number of a factor. The same criteria are also necessary for $\chi(G \times H)$ to differ significantly from $\min\{\chi(G), \chi(H)\}$.

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