THE CHROMATIC NUMBER OF THE PRODUCT OF 15-CHROMATIC GRAPHS CAN BE 14

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Abstract. We give an example of a 14-chromatic product of factors with chromatic numbers at least 15. Our example is an adaptation of those of Shitov and of Zhu, using graphs with minimal colourings that are “wide” in the sense of Simonyi and Tardos. If a certain graph with 13965 vertices has chromatic number at least 12, then the same argument produces an example of an 11-chromatic product of factors with chromatic numbers at least 12.

1. Introduction

The categorical product of two graphs $G$ and $H$ is the graph $G \times H$ with vertex-set $V(G \times H) = V(G) \times V(H)$, whose edges are the pairs $\{(g_1, h_1), (g_2, h_2)\}$ such that $\{g_1, g_2\}$ is an edge of $G$ and $\{h_1, h_2\}$ is an edge of $H$. Shitov [8] proved that the chromatic number of a categorical product of graphs can be smaller than the minimum of the chromatic numbers of the factors, hence disproving Hedetniemi’s conjecture of 1966 [6].

The short elegant argument was first adapted in [10, 5, 12] to show that the ratio $\min\{\chi(G), \chi(H)\}/\chi(G \times H)$ can be bounded away from 1. Out of necessity these investigations involved graphs with very large chromatic numbers. However along the way, Zhu noticed that the use of asymptotics in Shitov’s original argument could be avoided. This led to his construction of much smaller examples in [13]. Here we give a modification of the construction that also avoids the use of the injective colourings. Instead we use “wide colourings” in the sense of Simonyi and Tardos [9]. This allows to further decrease the chromatic number and size of the examples.

2. The example

For an integer $n$, the graph $G_n$ has for vertices all triplets $(x, Y, Z)$ such that $x \in \{1, \ldots, n\}$, $Y, Z \subseteq \{1, \ldots, n\}$, $x \in Z$, $Y \neq \emptyset$ and $Y \cap Z = \emptyset$. The edges of $G_n$ are the pairs $\{(x, Y, Z), (x', Y', Z')\}$ such that $x \in Y' \subseteq Z$, $Z \cap Z' = \emptyset$, $Z' \supseteq Y \supseteq x'$. The natural $n$-colouring $\gamma$ of $G_n$, defined by $\gamma(x, Y, Z) = x$. It is a proper colouring, thus $\chi(G_n) \leq n$.

Lemma 1 ([4, 9, 11]). $\chi(G_n) = n$.

In [4], our graph $G_n$ is $H(1, n, 3)$ in Theorem 4, in [9], it is $W(3, n)$ in Theorem 24, and in [11] it is $\Omega_2(K_n)$ in Theorem 1.1. Of course, the difficulty resides in proving that $G_n$ cannot be properly coloured with $n - 1$ colours. It is easy to find many proper $n$-colourings of $G_n$, by using a colour in the second or third coordinate.

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instead of the first. However, the natural colouring $\gamma$ has important properties that will be useful to us.

For $x \in \{1, \ldots, n\}$, we let $N(\gamma^{-1}(x))$ denote the *neighbourhood* of the $x$-th colour class, which consists of all vertices with a neighbour in $\gamma^{-1}(x)$. Similarly, $N(N(\gamma^{-1}(x)))$ consists of all vertices with a neighbour in $N(\gamma^{-1}(x))$, including $\gamma^{-1}(x)$ itself. We then have the following.

(i) The elements of $N(\gamma^{-1}(x))$ all have $x$ in their second coordinate, hence no two of them are adjacent.

(ii) The elements of $N(N(\gamma^{-1}(x)))$ all have $x$ in their third coordinate, hence no two of them are adjacent.

These properties contrast with the intuitive properties of a colour class in an optimal colouring: Since colour $x$ is necessary, shouldn’t $N(\gamma^{-1}(x))$ have chromatic number $n - 1$? Indeed in [1], Gyárfás, Jensen and Stiebitz mention a 1997 question to this effect by Harvey and Murty. This motivates their work on “graphs with strongly independent colour classes”, that is, those with an optimal colouring which satisfies property (i). The “wide colourings” of Simonyi and Tardos [9] are those which satisfy properties (i) and (ii). The development of topological bounds on chromatic numbers allowed to prove the compatibility of such properties with the optimality of a colouring.

We now focus on the graph $G_9$, more precisely on the lexicographic product $G_9[K_4]$. Its vertex-set is $V(G_9) \times V(K_4)$ (where $V(K_4) = \{1, 2, 3, 4\}$), and its edges are the pairs $\{(u, v), (u', v')\}$ such that $\{u, u'\}$ is an edge of $G_9$ or $u = u'$ and $\{v, v'\}$ is an edge of $K_4$.

**Lemma 2.** $\chi(G_9[K_4]) > 14$.

**Proof.** A 14-colouring of $G_9[K_4]$ would correspond to a homomorphism from $G_9$ to the Kneser graph $K(14, 4)$ (see [3]). However, $\chi(G_9) = 9$ by Lemma 1 while $\chi(K(14, 4)) = 8$ (see [7]).

This graph $G_9[K_4]$ is the first factor in our example. The second is the exponential graph $K_{14}^{G_9[K_4]}$. It has all functions from $V(G_9[K_4])$ to $V(K_{14})$ as vertices, with two of these, $f, g$, being adjacent if for every edge $\{(u, v), (u', v')\}$ of $G_9[K_4]$, we have $f(u, v) \neq g(u', v')$. It is well known that $G_9[K_4] \times K_{14}^{G_9[K_4]}$ admits a proper 14-colouring $c$ defined by $c((u, v), f) = f(u, v)$ (see [2]). Therefore to complete our example it only remains to prove the following.

**Lemma 3.** $\chi(K_{14}^{G_9[K_4]}) > 14$.

**Proof.** The argument follows the mold used in [8, 13]. Suppose that $\phi : K_{14}^{G_9[K_4]} \rightarrow K_{14}$ is a proper colouring. We can assume (possibly by permuting colours) that $\phi$ colours the constant maps each with its constant value. Therefore, for any vertex $f$ of $K_{14}^{G_9[K_4]}$, the colour $\phi(f)$ is in the image of $f$.

Let $f_\gamma$ be the vertex of $K_{14}^{G_9[K_4]}$ defined by $f_\gamma(u, v) = \gamma(u)$ (where $\gamma$ is the natural colouring of $G_9$). Without loss of generality, $\phi(f_\gamma) = 1$. For $i \in \{6, \ldots, 14\}$, we define the element $g_i$ of $K_{14}^{G_9}$ by

$$g_i(u, v) = \begin{cases} 
  v + 1 & \text{if } u \in N(N(\gamma^{-1}(1))), \\
  v + 5 & \text{if } u \in N(\gamma^{-1}(1)), \\
  i & \text{otherwise}.
\end{cases}$$
Then \( \{g_6, \ldots, g_{14}\} \) forms a 9-clique in \( K_{14}^{G_9[K_4]} \). Indeed suppose for a contradiction that there exist distinct values \( i, i' \) and adjacent vertices \( (u, v) \), \( (u', v') \) of \( G_9[K_4] \) such that \( g_i(u, v) = g_{i'}(u', v') \). This common value cannot be in \( \{10, \ldots, 14\} \) since these values are each in the image of only one of the functions. Neither can it be a value \( j \in \{2, 3, 4, 5\} \), since each of these values are taken by all the functions only on the independent set \( N(N(\gamma^{-1}(1))) \times \{j - 1\} \). Nor can it be a value \( j \in \{6, 7, 8, 9\} \), since each of these values are taken by all the functions on the independent set \( N(\gamma^{-1}(1)) \times \{j - 5\} \), with \( g_j \) taking it additionally on the set 
\[
(V(G_9[K_4]) \setminus (N(\gamma^{-1}(1)) \cup N(N(\gamma^{-1}(1)))) \times V(K_4),
\]
which is not joined to any vertex in \( N(\gamma^{-1}(1)) \times \{j - 5\} \).

Therefore \( \phi \) uses nine colours on \( \{g_6, \ldots, g_{14}\} \), thus some \( g_i \) is coloured with a colour not in \( \{2, \ldots, 9\} \). Since \( \phi(g_i) \) is in the image of \( g_i \), we then have \( i \in \{10, \ldots, 14\} \). Now consider the function \( h \) defined by 
\[
h(u, v) = \begin{cases} 
i \text{ if } u \in \gamma^{-1}(1) \cup N(\gamma^{-1}(1)), \\
1 \text{ otherwise.}
\end{cases}
\]
Then \( h \) is adjacent to \( f_7 \) since \( f_7 \) does not take the value \( i \) and takes the value 1 only on \( \gamma^{-1}(1) \times V(K_4) \). Similarly, it is adjacent to \( g_i \) since \( g_i \) does not take the value 1 and takes the value \( i \) only on the complement of \( N(\gamma^{-1}(1)) \cup N(N(\gamma^{-1}(1))) \times V(K_4) \). Thus \( h \) can neither be coloured 1 or \( i \), but these are the only options. This shows that the colouring \( \phi \) cannot exist.

\[\square\]

**Corollary 4.** The chromatic number of the product of 15-chromatic graphs can be 14.

**Proof.** The two 15-chromatic graphs can be taken as \( G_9[K_4] \) and \( K_{14}^{G_9[K_4]} \) or 15-chromatic subgraphs of these if they have a larger chromatic number. \[\square\]

### 3. Comments

With the same argument, for each \( m \geq 4 \), both \( G_{2m+1}[K_m] \) and \( K_{3m+2}^{G_{2m+1}[K_m]} \) have chromatic number greater than \( 3m + 2 \), while their product is \( (3m + 2) \)-colourable. Since \( \chi(G_7) = \chi(K(11, 3)) \), it is not clear whether the graph \( G_7[K_3] \) (with 13965 vertices) has chromatic number greater than 11. If so, then the chromatic number of the product of 12-chromatic graphs can be 11. However \( G_5[K_2] \) is 7-colourable, so this is the limit of what this construction can do.

In an email conversation with Yaroslav Shitov, Gábor Simonyi, Gábor Tardos, Marcin Wrochna and Xuding Zhu, it was noted that the subgraph \( H \) of \( K_{14}^{G_9[K_4]} \) used in the proof has only 141 vertices\(^1\). In contrast, \( G_9[K_4] \) itself has 226980 vertices. So, perhaps it is more natural to consider \( G_9[K_4] \) as a subgraph of \( K_{14}^2 \) rather than \( H \) as a subgraph of \( K_{14}^{G_9[K_4]} \). The intrinsic description of graphs that can play the role of \( H \) is outlined in [13]. Bounds on the chromatic number of their exponential graphs need not involve lexicographic products or wide colourings. Perhaps smaller examples can be found along this way.

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References


