CONLEY-ZEHNDER INDEX AND BIFURCATION OF FIXED POINTS OF HAMILTONIAN MAPS

YANXIA DENG AND ZHIHONG XIA

Abstract. We study the bifurcations of fixed points of Hamiltonian maps and symplectic diffeomorphisms. We are particularly interested in the bifurcations where the Conley-Zehnder index of a fixed point changes. The main result is that when the Conley-Zehnder index of a fixed point increases (or decreases) by one or two, we observe that there are several bifurcation scenarios. Under some non-degeneracy conditions on the one-parameter family of maps, two, four or eight fixed points bifurcate from the original one. We give a relatively detailed analysis of the bifurcation in the two dimensional case. We also show that higher dimensional cases can be reduced to the two dimensional case.

1. Introduction

In the 1960s, Arnold [1] conjectured about the number of fixed points of a Hamiltonian diffeomorphism. Specifically, let \((M, \omega)\) be a symplectic manifold, i.e. \(M\) is a smooth manifold equipped with a closed and nondegenerate 2-form \(\omega\). Then we can assign to each function

\[ H : M \times \mathbb{R} \to \mathbb{R}; \quad H(x, t) = H_t(x) \]

a vector field \(X_t\) on \(M\) defined by \(\omega(\cdot, X_t) = dH_t\). This vector field is called the (exact) Hamiltonian vector field associated with the (time-dependent) Hamiltonian \(H\). If \(M\) is compact, then the differential equation

\[ \frac{d}{dt} \varphi_{H,t}(x) = X_H(\varphi_{H,t}(x)) \]

with initial condition \(\varphi_{H,0}(x) = x\) defines a family of smooth diffeomorphisms of \(M\), which also preserves the symplectic structure, i.e. for each \(t \in \mathbb{R}\) we have \(\varphi_t^* \omega = \omega\). We call a symplectic diffeomorphism which can be obtained as the time-\(T\) map of a \(T\)-periodic Hamiltonian system a Hamiltonian diffeomorphism. Arnold conjectured that the number of fixed points of a Hamiltonian diffeomorphism is at least the same as the number of critical points of a certain smooth function on \(M\).

\textit{Date:} Revised, May 21, 2013.
There have been a lot of work concerning Arnold conjecture, for more history of the Arnold Conjecture we refer to E. Zehnder [2]. We especially would like to mention Conley and Zehnder’s work concerning the periodic solutions of Hamiltonian systems and the so-called Conley-Zehnder index of a nondegenerate symplectic path. In the proof of the Arnold Conjecture, the Conley-Zehnder index serves the same function as the Morse index in Morse theory.

Due to intrinsic nature of Hamiltonian systems, the bifurcation of fixed points in a typical one-parameter family of symplectic diffeomorphisms is complicated and much more interesting. The typical bifurcations for non-Hamiltonian systems, such as saddle-node, periodic doubling and Hopf bifurcations, are either no longer possible or are replaced by its Hamiltonian variants (cf. Meyer [10]). For example, saddle-nodes are now saddle-centers. Interestingly, there are many possible new bifurcations. The goal of this paper is to study these new bifurcation phenomena. In particular, we try to understand how fixed points change when their Conley-Zehnder indices change with respect to some bifurcation parameters.

In this paper we will briefly introduce the definition of Conley-Zehnder index, and study some bifurcation phenomena of fixed points related to this index. In section 2 we recalled the definition of Conley-Zehnder index for a nondegenerate symplectic path. Section 3 is our main part. In section 3.1 we studied the bifurcation of the two-dimensional case and the main result is that when the Conley-Zehnder index of a fixed point of a one-parameter family of Hamiltonian maps changes directly from 1 to 3, there are at least two fixed points bifurcate from the original one. Section 3.2 is a relatively detailed analysis of this bifurcation concerning its normal forms. Section 3.3 is about this bifurcation phenomenon in higher dimensional case, and we will see that it is essentially the same as the 2-dimensional case. Finally, in section 4 we study the bifurcation of fixed points when the Conley-Zehnder index of a fixed point increases by one, i.e. from 1 to 2. This is a relatively simpler bifurcation of fixed points compared with the case studied in section 3, and it can be reduced to a one-dimensional pitchfork bifurcation.

We finish this section by presenting a simple example to illustrate the types of bifurcations we are interested in this paper.

We consider Hamiltonian maps defined on two sphere $S^2$. Let $H: S^2 \to \mathbb{R}$ be a Morse function. Suppose $H$ has exactly two critical points, i.e. the maximum $p$ and the minimum $q$. Let $\varphi_t$ denote the corresponding Hamiltonian flow on $S^2$. Then $p, q$ are elliptic fixed points of the Hamiltonian diffeomorphism $\varphi_t$ for each $t$. We use $t$
as our bifurcation parameter. When $t$ is small, it can be computed, as we shall see later, that $p, q$ has Conley-Zehnder index -1 and 1, respectively. This simply means that, nearby $p$ and $q$, the map slightly rotates one clockwise and the other one counterclockwise. If we "blow up" $p$ and $q$, replace them with boundary circles $C_p$ and $C_q$, we will get an area-preserving diffeomorphism of the annulus. The two boundary components rotate in the same direction.

We can also see it as follows: the fixed points of $\varphi_t$ coincides with the critical points of $H$ when $t$ is small, and $H$ has no other critical points than $p, q$, the diffeomorphism on the annulus should rotate the two boundary circles in the same direction. For otherwise, by Poincare-Birkhoff theorem, there should be at least two more fixed points.

On the other hand, if we increase time parameter $t$ to $t^*$ such that one of boundary components, say $C_q = \mathbb{R}/\mathbb{Z}$, progresses a little over 1, then the Conley-Zehnder index of $q$ goes from 1 to 3 (or from -1 to -3 depending on the orientation). Suppose that the index of $p$ stays the same at $t^*$ (this is true for a typical Hamiltonian function $H$). This way, we end up with an area-preserving map of the annulus, with one boundary components progresses more than 1 and the other boundary components progresses less than 1. By Poincare-Birkhoff theorem, we conclude that $\varphi_{t^*}$ should have at least two more fixed points, both with rotation number 1. We conclude that there are at least two more fixed points when Conley-Zehnder index of one of the fixed points changes from 1 to 3.

This is a motivating example. The general situation is more complicated, as we shall discuss.

For a Hamiltonian diffeomorphism close to identity, Arnold conjecture can be proved using a generating function and Morse theory. The Conley-Zehnder index is simply the Morse index except with a uniform shift in dimensions. The Morse index is always bounded by the dimension of a finite dimensional manifold. However, for the maps far away from identity, the Conley-Zehnder index of a fixed point can be arbitrarily large in both positive and negative directions. Naturally, we would like to understand how the index of the fixed points increases. This is the main objective of this paper.

2. CONLEY-ZEHNDER INDEX AND THE SYMPLECTIC GROUP

The Conley-Zehnder index assigns an integer to each nondegenerate symplectic path which starts at the identity matrix and ends at a
matrix which does not admit 1 as an eigenvalue. There are many references concerning the definition of this index. It was first introduced in Conley and Zehnder’s celebrated paper [4] in 1984. D.Salamon and E.Zehnder [5] developed this index later in 1992. A 60-page note by Jean Gutt [6] which can be found on Arxiv is a good reference for those who first hear of this index. In this paper, we will follow the exposition in [7].

Let \( Sp(2n) \) be the symplectic group,

\[
Sp(2n) = \{ A \in \mathbb{R}^{2n \times 2n} | A^TJA = J \}
\]

where

\[
J = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}
\]

such that \((\mathbb{R}^{2n}, \omega_0)\) with \(\omega_0(v, w) := Jv \cdot w\) the standard symplectic vector space. Here \(v \cdot w\) is the usual inner product on \(\mathbb{R}^{2n}\). Let us denote \(S^*(2n) := \{ \gamma \in C^0([0, 1], Sp(2n)) | \gamma(0) = I, \det(\gamma(1) - I) \neq 0 \} \) the space of all the nondegenerate symplectic path. The Conley-Zehnder index will be defined as the degree of a map into the unit circle for each element in \(S^*(2n)\). If we equip \(S^*(2n)\) with the compact-open topology, then the Conley-Zehnder index has a one-one correspondence with the connected component of \(S^*\).

First we study the topology of the symplectic group. Since \(Sp(2n)\) is a closed subgroup of \(GL(2n, \mathbb{R})\), it inherits a structure of Lie group. The following proposition can be found in [7].

**Proposition 2.1.** The symplectic group \(Sp(2n)\) is homeomorphic to \(\mathbb{R}^{n(n+1)} \times U(n)\), where \(U(n)\) is the unitary group.

The unitary group \(U(n)\) is isomorphic (as a Lie group) to the product of \(S^1\), the unit circle, and the special unitary group \(SU(n)\). Indeed, an explicit isomorphism is the following:

\[
\mathbb{R}/2\pi \mathbb{Z} \times SU(n) \to U(n) : (\theta, U) \mapsto e^{i\theta}U
\]

Therefore, \(Sp(2n)\) is homeomorphic to \(\mathbb{R}^{n(n+1)} \times S^1 \times SU(n)\). It is well known that \(SU(n)\) is simply connected, so the map \(\det : U(n) \to S^1\) induces an isomorphism of fundamental groups. Moreover, \(\pi_1(Sp(2n)) = \mathbb{Z}\).

There is a natural continuous function from \(Sp(2n)\) to \(S^1\) which induces an isomorphism of their corresponding fundamental groups, which we call the rotation function. To see the definition of this function,
we first study the eigenvalues and generalized eigenspaces of symplectic matrices, and the Krein signature on $Sp(2n)$. It is useful to introduce the Hermitian form

$$g(v, w) := \langle Gv, w \rangle, \quad G := -iJ$$

Where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product on $\mathbb{C}^{2n}$. The complex symplectic group $Sp(2n, \mathbb{C})$ consists of the $g$-unitary complex linear automorphisms of $\mathbb{C}^{2n}$. Equivalently, $A \in Sp(2n, \mathbb{C})$ if and only if $A^*GA = G$, where $A^* = A^T$. A matrix belongs to $Sp(2n)$ if and only if it is in $Sp(2n, \mathbb{C})$ and it is real. There are certain observations of the eigenvalues and the generalized eigenspaces (denoted by $E_\lambda$) of symplectic matrices.

1. If $A \in Sp(2n, \mathbb{C})$ has an eigenvalue $\lambda$, then it also has the eigenvalue $\bar{\lambda}^{-1}$, and $A$ has the same Jordan form on $E_\lambda$ and $E_{\bar{\lambda}^{-1}}$.

2. The generalized eigenspaces $E_\lambda$ and $E_\mu$ are $g$-orthogonal, provided $\lambda \mu \neq 1$.

3. Denote by $\sigma(A)$ the spectrum of $A$, we can arrange the spectral decomposition of $A \in Sp(2n, \mathbb{C})$ in the following way:

$$\mathbb{C}^{2n} = \bigoplus_{\lambda \in \sigma(A), |\lambda| \geq 1} F_\lambda$$

where $F_\lambda := E_\lambda$ if $|\lambda| = 1$ and $F_\lambda := E_\lambda \oplus E_{\bar{\lambda}^{-1}}$ if $|\lambda| > 1$. From 2 above, we know the decomposition is $g$-orthogonal. Therefore, $g$ must be nondegenerate on each of the space $F_\lambda$.

4. Let $\lambda$ be an eigenvalue of $A \in Sp(2n, \mathbb{C})$ outside the unit circle, with algebraic multiplicity $d$. Then $g(v, v) = 0$ for every $v \in E_\lambda$. Also $g$ restricted to the $2d$-dimensional space $F_\lambda$ has a $d$-dimensional isotropic subspace, and we conclude that $g$ has signature $(d, d)$ on $F_\lambda$.

5. The Hermitian form $g$ may have any signature on $F_\lambda = E_\lambda$ in the case $|\lambda| = 1$, and in this case, we define the Krein signature of $\lambda$ to be the signature of the restriction of the Hermitian form $g$ on $E_\lambda$.

6. Now let $A$ be a real symplectic matrix, we know it has eigenvalues appear in quadruples as $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$. If $\lambda$ is an eigenvalue of $A$ on the unit circle and has Krein signature $(p, q)$, then the eigenvalue $\bar{\lambda}$ has Krein signature $(q, p)$.

7. From 6 above, if $2k$ is the total algebraic multiplicity of the eigenvalues of $A$ on the unit circle, we can say that there are $\lambda_1, \cdots, \lambda_k$ Krein positive eigenvalues and $\bar{\lambda}_1, \cdots, \bar{\lambda}_k$ Krein negative eigenvalues.
Now we are ready to define the rotation function we have mentioned above.

**Definition 2.2.** Let $\lambda_1, \cdots, \lambda_{2n}$ be the eigenvalues of $A \in Sp(2n)$, repeated according to their multiplicity. The eigenvalue $\lambda_i$ is said of the first kind if either $|\lambda_i| < 1$ or $|\lambda_i| = 1$ and $\lambda_i$ is Krein-positive. Otherwise, it is said of second kind.

This definition is from [7]. Therefore, we can always order the eigenvalues of $A$ as $\lambda_1, \cdots, \lambda_n, \lambda_1^{-1}, \cdots, \lambda_n^{-1}$ where $\lambda_1, \cdots, \lambda_n$ are of the first kind. With such an ordering, the rotation function $\rho : Sp(2n) \to S^1$ is defined as

$$\rho(A) := \prod_{i=1}^{n} \frac{\lambda_i}{|\lambda_i|}$$

Equivalently,

$$\rho(A) = (-1)^m \prod_{\mu \in \sigma(A) \cap S^1 \setminus \{\pm 1\}} \mu^{p(\mu)}$$

where $2m$ is the total multiplicity of the real negative eigenvalues of $A$ and $(p(\mu), q(\mu))$ is the Krein signature of the eigenvalue $\mu \in S^1$.

**Theorem 2.3.** The rotation function $\rho$ defined above has the following properties:

(1) (Continuity) The function $\rho$ is continuous.
(2) (Symplectic invariance) $\rho(MAM^{-1}) = \rho(A)$ for any $A, M \in Sp(2n)$.
(3) (Splitting) If $(\mathbb{R}^{2n}, \omega_0) = (\mathbb{R}^{2m}, \omega_0) \oplus (\mathbb{R}^{2(n-m)}, \omega_0)$, and if

$$A = \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right)$$

with $A_1 \in Sp(2m)$ and $A_2 \in Sp(2(n - m))$, then $\rho(A) = \rho(A_1)\rho(A_2)$
(4) (Value on the unitary group) If $A \in Sp(2n) \cap O(2n)$ and $U$ is its corresponding matrix in $U(n)$, then $\rho(A) = \det(U)$.
(5) (Normalization) $\rho(A) = \pm 1$ if $A$ does not have eigenvalues on the unit circle.
(6) (Homotopy) The map $\rho$ induces an isomorphism of fundamental groups.

The proof of this theorem can be found in [5] [6] and [7]. Also, the first five properties characterize the family of functions $\rho$. We will represent another important theorem before we give the definition of
Conley-Zehnder index.

Decompose $Sp(2n)$ into three subsets:

$$Sp(2n)^+ = \{ A \in Sp(2n) | \det(A - I) > 0 \}$$
$$Sp(2n)^- = \{ A \in Sp(2n) | \det(A - I) < 0 \}$$
$$Sp(2n)^0 = \{ A \in Sp(2n) | \det(A - I) = 0 \}$$

Set $Sp(2n)^* := Sp(2n)^+ \cup Sp(2n)^-$, and we fix two representatives $W^\pm$ in $Sp(2n)^\pm$ with $W^+ = -I$ and $W^- = \text{diag}(2, 1/2, -1, \cdots, -1)$. It is easy to compute that $\rho(W^+) = (-1)^n$ and $\rho(W^-) = (-1)^{n-1}$.

**Theorem 2.4.** The open sets $Sp(2n)^+$ and $Sp(2n)^-$ are connected and contractible.

For the proof of this theorem we refer to [5] [6] and [7].

Now given a path $\gamma \in S^*(2n)$, we may extend it to a continuous path $\tilde{\gamma} : [0, 2] \to Sp(2n)^*$ such that $\tilde{\gamma}$ coincide with $\gamma$ on the interval $[0, 1]$, and $\tilde{\gamma}(t) \in Sp(2n)^*$ for all $t \geq 1$ and $\tilde{\gamma}(2) \in \{W^\pm\}$. Note that $\rho(W^+) = (-1)^n$ and $\rho(W^-) = (-1)^{n-1}$, thus the square $(\rho \circ \tilde{\gamma})^2 : [0, 2] \to S^1$ is a loop.

**Definition 2.5 (Conley-Zehnder index).** Given the notation as above, the Conley-Zehnder index of a path $\gamma \in S^*(2n)$ is the integer given by the degree of the map $(\rho \circ \tilde{\gamma})^2 : [0, 2] \to S^1$.

$$\mu_{\text{CZ}}(\gamma) = \deg(\rho \circ \tilde{\gamma})^2$$

Note that since the open sets $Sp(2n)^+$ and $Sp(2n)^-$ are connected and contractible, the Conley-Zehnder index of $\gamma$ does not depend on the choice of the extension $\tilde{\gamma}$. Also, it is not difficult to see that two nondegenerate symplectic paths $\gamma_0$ and $\gamma_1$ are in the same connected component of $S^*(2n)$ if and only if they have the same Conley-Zehnder index. That is, the Conley-Zehnder index is invariant under homotopy as long as the end point of the symplectic path stays in one of the connected components of $Sp(2n)^\pm$.

We can also define Conley-Zehnder index for a periodic linear Hamiltonian system. Consider the one-periodic case. A linear one-periodic Hamiltonian system in $\mathbb{R}^{2n}$ has the form:

$$\dot{Z}(t) = JS(t)Z(t)$$

where $S(t)$ is a one-periodic path of symmetric matrices. Let $\gamma(t)$ be the principal fundamental solution of the equation, then $\gamma(t)$ is symplectic for each $t$. The periodic Hamiltonian system is called nondegenerate if
1 is not a Floquet multiplier. i.e. det(γ(1)−I) ̸= 0. We define the C-Z index for the nondegenerate linear periodic Hamiltonian system to be the C-Z index of its principal fundamental solution.

As a special case, consider a time independent Hamiltonian \( H(x) \), which is also a Morse function on \( M \). Then every critical point \( x^* \) of \( H \) is a stationary solution of \( \dot{x}(t) = X_H(t, x(t)) \). By Floquet theory, we can linearize the equation at \( x^* \), which becomes: \( \dot{x}(t) = Jd^2H(x^*)x(t) \). The fundamental solution is the exponential path \( \exp(tJS) \), where \( S = d^2H(x^*) \). If \( |S| \) is sufficiently small, we can get the relation that \( \mu_{CZ}(x^*) = n - \mu^-(S) \), where \( \mu^-(S) \) is the number of negative eigenvalues of \( S \). We can see this reduces to the classical Morse theory.

The Conley-Zehnder index somehow describes how many times the eigenvalues of the matrix of the symplectic path crosses 1. This can be seen from an example of two dimensional exponential path \( \gamma = \exp(tJS) : [0, T] \to Sp(2) \), with \( S = \text{diag}(\lambda_1, \lambda_2) \). Let \( \text{Sig}S \) denote the signature of \( S \), i.e. the number of positive eigenvalues minus the number of negative eigenvalues. Then,

\[
\mu_{CZ}(\gamma, T) = \text{Sig}S\left(\frac{1}{2} + \left\lfloor \frac{\sqrt{\lambda_1\lambda_2}T}{2\pi} \right\rfloor\right),
\]

if \( \text{Sig}S \neq 0 \); and it is equal to zero when \( \text{Sig}(S) = 0 \). The computation can be found in [6]. Actually, when \( \lambda_1, \lambda_2 > 0 \), we have:

\[
\gamma(t) = \begin{pmatrix}
\cos \sqrt{\lambda_1\lambda_2}t & -\frac{2\pi}{\lambda_1} \sin \sqrt{\lambda_1\lambda_2}t \\
\frac{2\pi}{\lambda_2} \sin \sqrt{\lambda_1\lambda_2}t & \cos \sqrt{\lambda_1\lambda_2}t
\end{pmatrix}
\]

with eigenvalues \( e^{\pm i\sqrt{\lambda_1\lambda_2}t} \). And \( e^{i\sqrt{\lambda_1\lambda_2}t} \) is the Krein positive one. The Conley-Zehnder index will change by 2 whenever \( t \) passes through an integer multiples of \( \frac{2\pi}{\sqrt{\lambda_1\lambda_2}} \), i.e. when the eigenvalues of the matrix \( \exp(tJS) \) passes through one. Moreover, we can see that whenever the time \( t \) increases by \( \frac{2\pi}{\sqrt{\lambda_1\lambda_2}} \), the absolute value of the Conley-Zehnder index will increase. This property somehow indicates that Conley-Zehnder index can describe the “rotation” of the flow along a periodic solution.

### 3. Bifurcation when C-Z index changes from 1 to 3

In this section, we study this bifurcation phenomenon of one-parameter Hamiltonian diffeomorphisms when the Conley-Zehnder index of a fixed
point changes from 1 to 3. As we noted in the introduction, this is how
the Conley-Zehnder index can increase indefinitely even for finite di-

3.1. Two-dimensional Bifurcation. Suppose \((M, \omega)\) is a compact
two-dimensional symplectic manifold.

\[ F_\mu = F(\mu, \cdot) : \mathbb{R} \times M \to M \]
is a one-parameter family of Hamiltonian maps on \((M, \omega)\).

The Conley-Zehnder index of a fixed point of a Hamiltonian map is
defined by considering the fixed point as a rest point of the correspond-
ing Hamiltonian equations, and the Conley-Zehnder index is obtained
from the linearized equation at this rest point. We want to study the
bifurcation of fixed points of \(F_\mu\) when the Conley-Zehnder index of a
fixed point changes directly from 1 to 3.

Assume \(p \in M\) is a fixed point of \(F_\mu\) for all the \(\mu \in \mathbb{R}\). Locally
we can regard \(F_\mu\) as an area-preserving map of a neighborhood of the
origin 0 in \(\mathbb{R}^2\):

\[ F_\mu : (x, y) \mapsto (X_\mu(x, y), Y_\mu(x, y)) \]

And 0 is an isolated fixed point for \(F_\mu\) in this neighborhood. Since
\(F_\mu\) is symplectic, \(|DF_\mu(0)| = 1\), the eigenvalues of \(DF_\mu(0)\), which is
called the multipliers of the fixed point 0, are of the form \(\lambda, \lambda^{-1}\). The
fixed point is called hyperbolic (elliptic) if its multipliers \(\lambda, \lambda^{-1}\) satisfy
\(|\lambda| \neq 1\) (resp.\(|\lambda| = 1\) and \(\lambda \neq \pm 1\)). Suppose the Conley-Zehnder index
of the origin changes from 1 to 3 when \(\mu\) passes 0, then the eigenvalue
of \(DF_\mu(0)\) should be on the unit circle and passes through 1. By a
re-parametrization, the eigenvalues of \(DF_\mu(0)\) could be assumed to be
\(e^{\pm i\mu}\), with \(e^{i\mu}\) the Krein-positive one. When \(\mu\) passes zero, the Krein-
positive eigenvalue \(e^{i\mu}\) passes 1, as indicated in the following picture:
Note that the case when a pair of eigenvalues on the unit circle passing through 1 and moving to the real line will corresponds to Conley-Zehnder index changing by 1. But this is not the “rotation” increasing case, instead, it will indicate that the fixed point change from elliptic to hyperbolic. In the last section we will analyse the bifurcation of fixed points when the Conley-Zender index changes from 1 to 2.

The following lemma will transfer our bifurcation question of the fixed points of one-parameter hamiltonian diffeomorphisms to the bifurcation of critical points of generating functions.

**Lemma 3.1.** Suppose $F_\mu$ is a one-parameter family of symplectic diffeomorphisms of an open disk centered at $x = y = 0$ in the $(x, y)$ plane into the plane. Assume $(0, 0)$ is an isolated fixed point of $F_\mu$, for $|\mu| < 1$ in the disk. And

$$DF_\mu(0) = \begin{pmatrix} \cos \mu & -\sin \mu \\ \sin \mu & \cos \mu \end{pmatrix}$$

Then there is an appropriate generating function $g$ of $F_\mu$ such that its critical points correspond to the fixed points of $F_\mu$, and $g$ has the form $g(\mu, \xi, \eta) = 1/2(\xi^2 + \eta^2) + \text{h.o.t.}$, where h.o.t. denotes higher order terms.

**Proof.** First let’s forget about the parameter $\mu$ and consider a symplectic map $F$. Since it is symplectic, there exists a generating function $g$ such that:

$$dg = (Y - y)d(X + x) - (X - x)d(Y + y)$$

When $-1$ is not an eigenvalue of $DF(0)$, we can introduce new variables $\xi = X + x$ and $\eta = Y + y$, so that:

$$\frac{\partial g}{\partial \xi} = Y - y, \frac{\partial g}{\partial \eta} = x - X$$

Moreover, the critical points of $g$ are in 1-1 correspondence with fixed points of $F$ by the above relationship. Since by assumption $(0, 0)$ is our fixed point, we can write $g(\xi, \eta) = g_2(\xi, \eta) + \text{h.o.t.}$ Assume the quadratic part $g_2(\xi, \eta) = \frac{1}{2}(\alpha \xi^2 + 2\beta \xi \eta + \gamma \eta^2)$, then

$$d^2g(0) = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

From the relation of $g$ and $F$, we can solve for $DF(0)$:

$$\begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix}^{(0, 0)} = \frac{1}{1 + \Delta} \begin{pmatrix} (\beta - 1)^2 - \alpha \gamma & -2\gamma \\ 2\alpha & (\beta + 1)^2 - \alpha \gamma \end{pmatrix}$$
where $\Delta = \alpha \gamma - \beta^2$. Now, since $F_\mu$ is depending on a parameter, the corresponding generating function also depends on a parameter, denoted by $g(\mu, \xi, \eta)$. Suppose the quadratic part of $g(\mu, \xi, \eta)$ is equal to

$$\frac{1}{2}(\alpha(\mu)\xi^2 + 2\beta(\mu)\xi\eta + \gamma(\mu)\eta^2),$$

by our assumption of $DF_\mu(0)$, we can easily calculate that the quadratic part of $g$ is equal to

$$\frac{1}{2}\frac{\sin \mu}{1 + \cos \mu}(\xi^2 + \eta^2),$$

hence we get the claim in the lemma for small $\mu$. □

Note that the trace of the matrix $DF(0)$ will indicate the stability type of the origin. Denote the trace by $\tau$, then when $|\tau| > 2$, the origin is a hyperbolic fixed point, and when $|\tau| < 2$ it is an elliptic fixed point. In the above lemma, we have $\tau = 2(1 - \Delta)(1 + \Delta)^{-1}$ where $\Delta$ is the determinant of the Hessian of the generating function at the origin, i.e. $d^2 g(0)$. Hence, a hyperbolic fixed point of $F$ corresponds to a saddle point of the function $g$, and an elliptic fixed point corresponds to a maximum or minimum of the function $g$. Therefore, we can transfer our question to the study of the bifurcation of critical points of $g_\mu$.

The above lemma tells us that the origin changes from a local maximum to a local minimum as $\mu$ varies from negative to positive for the function $g$. We know then there must be some bifurcations of critical points. This simply follows from Morse theory, or more generally, Conley index theory for isolated invariant set [8]. Before we give a detailed analysis of these bifurcations, we would like to present a simple topological version of the bifurcations using Conley index. Roughly speaking, an isolated invariant set of a differential equation has an index which is the homotopy type of a pointed space. A set in the phase space is called invariant if it is the union of solution curves. It is isolated if it is the maximal invariant set in some neighborhood of itself. A compact such neighborhood is called an isolating neighborhood for the invariant set. The index of an isolated invariant set is then obtained by collapsing the exit set of its isolating neighborhood. An important property of isolating neighborhood is the continuation property: if $N$ is an isolating neighborhood of some equation, then $N$ will be an isolating neighborhood for all equations near the given one. The isolated invariant sets thus determined by $N$ are the continuations, and they should have the same index.
The gradient equation of $dz/dt = -\nabla g(\mu, z)$ has the origin as an isolated rest point for all values of $\mu$. Furthermore, it is a repeller when $\mu$ negative, and an attractor when $\mu$ positive. The index of the repeller is a pointed 2-sphere, and the index of the attractor is a pointed 0-sphere. Thus for small $\mu$ that passes zero, there must be a bifurcation of rest points, otherwise the isolating neighborhood will define a continuation for some small negative $\mu$ and positive $\mu$ which is impossible as the origin has different index. Specifically, if we further assume the origin is also a local maximum for $g_\mu$ at $\mu = 0$, and all the rest points are nondegenerate, we can conclude that there must be at least two new rest points bifurcate from the origin.

**Lemma 3.2.** Suppose $g(\mu, z)$ is a one-parameter smooth function on $\mathbb{R}^n$ such that the origin is an isolated critical point for all $\mu \in [-1, 1]$. And it is a local maximum for $\mu \leq 0$, a local minimum for $\mu > 0$. Then there are at least two critical points bifurcate from the origin when the critical points of $g$ are nondegenerate.

**Proof.** This is a simple result of the Conley index theory. Consider the one-parameter gradient equation

$$\frac{dz}{dt} = -\nabla g(\mu, z)$$

By assumption the origin is an isolated rest point, and it is a repeller for $\mu \leq 0$, and an attractor for $\mu > 0$. Let $N_0$ be the isolating neighborhood of the origin when $\mu = 0$, for example, we can take a small disk centered at the origin. Thus the boundary circle of $N_0$ is the exit set, and we see that its index is a pointed $n$-sphere $(S^n, \cdot)$. When $\mu < 0$, since the index of the origin is the same as the invariant set of $N_0$, we cannot get more information. On the other hand, when $\mu > 0$, the index of the origin is a pointed 0-sphere, which is different from the invariant set of $N_0$. Since the origin is an isolated rest point by assumption, we can take a smaller disk $D_\mu$ centered at the origin such that it is an isolating neighborhood. Then $N_\mu = N_0 \setminus D_\mu$ is also an isolating neighborhood, and by collapsing its exit set, the index of its invariant set is $(S^n \vee S^1, \cdot)$. This index has betti number $\beta_1 = 1, \beta_n = 1$. Thus, when the critical points are nondegenerate, the Morse theorem tells us there are at least two more rest points. \qed

Therefore, from the above discussion we have proved the following:

**Theorem 3.3.** Suppose $F_\mu$ is a one-parameter family of Hamiltonian maps from an open disk centered at the origin of the plane to the plane, and the origin is a fixed point of $F_\mu$ for all $\mu \in [-1, 1]$. Suppose the
Conley-Zehnder index of the origin changes from 1 to 3 as $\mu$ passes from negative to positive, and the origin does not have other C-Z index than 1 and 3 for all small $|\mu|$. Suppose all the fixed points of $F_{\mu}$ are nondegenerate, then there must be at least two fixed points bifurcate from the origin as $\mu$ passes zero.

Note that the origin does not have other Conley-Zehnder index than 1 and 3 for all small $|\mu|$ indicates that when $\mu = 0$, the fixed point should have the same type as the fixed point when $\mu < 0$ or $\mu > 0$.

3.2. Normal form and bifurcation of the two-dimensional case.

The above theorem is a rough topological result. It lacks the analytical details on how the bifurcation occurs. We now try to derive a more precise formulation of this phenomenon. We will assume the origin is a local maximum when the parameter $\mu = 0$. Our function is $g(\mu, x, y) = \frac{1}{2} \mu (x^2 + y^2) + h.o.t.$, and we require $-g(0, x, y)$ to be positive definite so that the origin is a maximum when $\mu = 0$. First we need to analyze the normal form of $g(\mu, x, y)$ under our condition.

Since the third order homogeneous polynomials $ax^3 + bx^2y + cxy^2 + dy^3$ can never be definite, our function should start from the fourth order at $\mu = 0$. Therefore, our function is as follows:

$$g(\mu, x, y) = \frac{1}{2} \mu (x^2 + y^2) + (a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4)$$

$$\quad + \mu(b_0 x^3 + b_1 x^2 y + b_2 xy^2 + b_3 y^3) + h.o.t.$$  

From our condition that the origin is an isolated maximum when $\mu = 0$, it is not difficult to deduce that $a_0 < 0, a_4 < 0$. Also we can find a coordinate transformation such that the parameter $a_1 = 0$. As a matter of fact, we can take $x = u + av$, $y = v - au$, for some $a \in \mathbb{R}$ depending on the coefficients of the $a_i's$. This will not affect the form of the quadratic term $1/2 \mu (x^2 + y^2)$. From now on, we write our function as

$$g(\mu, x, y) = \frac{1}{2} \mu (x^2 + y^2) - (x^4 + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4)$$

$$\quad + \mu(\mathcal{O}((|x| + |y|)^3)) + h.o.t.$$  

Here we write $\mu(b_0 x^3 + b_1 x^2 y + b_2 xy^2 + b_3 y^3)$ as $\mu(\mathcal{O}((|x| + |y|)^3))$, and we will see later that this term turns out to be a higher order term and it will not affect the existence of the critical points of $g(\mu, x, y)$ for nondegenerate cases.
3.2.1. Nondegenerate case with $a_3 \neq 0$. A quartic polynomial

$$g(x, y) = x^4 + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$$

is positive definite if and only if the coefficients satisfy one of the following conditions:

1. $\Delta > 0$, where $\Delta = (a_4 + 3a_2^2)^3 - 27(a_2a_4 - a_3^2 - a_2^3)^2$, and in addition, one of the following two cases
   (a) $a_2 \geq 0$, $a_2a_4 \geq a_3^2$.
   (b) $a_2 < 0, 9a_2^2 - a_4 < 0$, $12(a_2a_4 - a_3^2 - a_2^3)(a_4 + 3a_2^2) < 0$.

2. $\Delta = 0$, $a_3 = 0$, $a_2 > 0$. This is the case that $g(x, y)$ is a perfect square with $g(x, y) = (x^2 + 3a_2y^2)^2$.

Positive definiteness of $g(x, y)$ is equivalent to that the quartic equation $x^4 + 6a_2x^2 + 4a_3x + a_4 = 0$ has no real roots. Wang and Qi [9] gave a necessary and sufficient condition for general quartic polynomial $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$. We refer to [9] for further details.

Under these conditions for positive definiteness, we take a further look at our bifurcation problem. Our function is

$$g(\mu, x, y) = \frac{1}{2}\mu(x^2 + y^2) - (x^4 + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4)$$

$$+ \mu(\mathcal{O}((|x| + |y|)^3))$$

To find its critical points, we solve $\nabla g = 0$, where

$$\nabla g = (g_x, g_y)$$

$$= \left( \frac{\mu x - 4x^3 - 12a_2xy^2 - 4a_3y^3}{\mu y - 12a_2x^2y - 12a_3xy^2 - 4a_4y^3} \right) + \mu(\mathcal{O}((|x| + |y|)^2))$$

$a_4 > 0$ always hold by the positive definiteness condition, and we only consider the generic case when $a_3 \neq 0$. In fact, the degenerate case with $a_3 = 0$ can be easily analysed. In this case we will have $a_2 \neq 0$.

First, it is easy to see that $\nabla g = 0$ implies that $\mu$ is of second order on $x$ and $y$, i.e. $\mu = \mathcal{O}((|x| + |y|)^2)$. Thus, $\mu(\mathcal{O}((|x| + |y|)^3))$ in $g(\mu, x, y)$ is a higher order term: $\mathcal{O}((|x| + |y|)^5)$. That is, to analyze the number of critical points of $g(\mu, x, y)$, we can, for simplicity, safely ignore the $\mu(\mathcal{O}((|x| + |y|)^3))$. As long as the fixed points we find are non-degenerate, the result extends to the full system for small values of $x$ and $y$. 
The following analysis is carried out with only the dominant part of $g(\mu, x, y)$. We obtain, dropping higher order terms,

$$g(\mu, x, y) = \frac{1}{2}\mu(x^2 + y^2) - (x^4 + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4).$$

Then to solve

$$\nabla g = \left( \begin{array}{c} \mu x - 4x^3 - 12a_2xy^2 - 4a_3y^3 \\ \mu y - 12a_2x^2y - 12a_3xy^2 - 4a_4y^3 \end{array} \right) = 0$$

we have the following:

(a) When $y = 0$, we have $\mu x - 4x^3 = 0$, i.e. $x = 0$ or $\mu = 4x^2$;

(b) When $x = 0$, we have $-4a_3y^3 = 0$, which implies $y = 0$.

(c) When $xy \neq 0$, we can take $yg_x - xg_y = 0$, which implies

$$(1 - 3a_2)x^3 - 3a_3x^2y + (3a_2 - a_4)xy^2 + a_3y^3 = 0$$

Since $xy \neq 0$, we divide by $y^3$ and denote by $w = \frac{x}{y}$, get

(1) 

$$\left(1 - 3a_2\right)w^3 - 3a_3w^2 + (3a_2 - a_4)w + a_3 = 0$$

For each real root of $w$, we will have $x = wy$ and plug into the equation $g_y = 0$ which gives us

(2) 

$$y[\mu - 4(3a_2w^2 + 3a_3w + a_4)y^2] = 0$$

In this way, we solve the gradient equation $\nabla g = 0$.

Now our problem is to determine under what conditions of the parameters $a_2, a_3, a_4$ will the cubic equation (1) have one real root, three real roots etc. The discriminant of a cubic equation $ax^3 + bx^2 + cx + d = 0$ is equal to $\Delta_3 = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$, with the following:

- $\Delta_3 > 0$, has 3 real distinct roots;
- $\Delta_3 = 0$, has a double root, and all roots are real;
- $\Delta_3 < 0$, has only one real root and two complex conjugate roots.

By computation, the discriminant of the cubic equation (1) is equal to

$$- 54a_3^2(1 - 3a_2)(3a_2 - a_4) + 108a_3^4 + 9a_2^2(3a_2 - a_4)^2 - 4(1 - 3a_2)(3a_2 - a_4)^3 - 27(1 - 3a_2)^2a_3^2$$

If we denote by

$$\alpha = \frac{3a_2 - 1}{3a_3}, \quad \beta = \frac{a_4 - 3a_2}{3a_3},$$

then

$$\Delta_3 = (3a_3)^4[-3\alpha^2 - (6\beta + 4\beta^3)\alpha + \left(\frac{4}{3} + \beta^2\right)]$$

From here we can see that the discriminant $\Delta_3$ of equation (1) can have all three possible signs, i.e. negative, zero, or positive. This implies
that the function \( g(\mu, x, y) \) has 4, 6, or 8 critical points bifurcate from the origin when \( \mu > 0 \), with the possibility that some of them may not be open case.

Furthermore, we take the positive definiteness condition of the parameters into account. And we only consider the case of \( \Delta > 0 \), where \( \Delta = (a_4 + 3a_2^2)^3 - 27(a_2a_4 - a_3^2 - a_2^2)^2 \). Under this condition, we know there are two subcases of the parameters, that is

(a) \( a_2 \geq 0, a_2a_4 \geq a_3^2 \). Or,

(b) \( a_2 < 0, 9a_2^2 - a_4 < 0, 12(a_2a_4 - a_3^2)^2 - a_4^2(a_4 + 3a_2^2) < 0 \).

One nice thing with case (a) is that the value \( 3a_2w^2 + 3a_3w + a_4 \) in equation (2) is always positive for all \( w \in \mathbb{R} \). That is, for each real root \( w \) we get from (1), we will have:

\[
\mu = 4(3a_2w^2 + 3a_3w + a_4)y^2
\]

Which from the relation of \( x = wy \) will give us

\[
\mu = 4(3a_2w^2 + 3a_3w + a_4)(1/w^2)x^2
\]

To determine the number of critical points created when \( \mu > 0 \), we only need to know the sign of \( \Delta_3 \) under this positive definiteness condition.

For case (b), the value \( 3a_2w^2 + 3a_3w + a_4 \) may be negative at first glance, but when taking a little analysis one would see that under our assumption of definiteness at \( \mu = 0 \), there should be no other critical points than the origin when \( \mu \leq 0 \). Thus, similar as case (a), the number of critical points created when \( \mu > 0 \) is determined by the sign of \( \Delta_3 \).

Unfortunately, in either cases, it seems to be complicated to get an analytical relation for the sign of \( \Delta_3 \). In fact, one should not expect to have a simple relation.

We use Mathematica to draw the regions in parameter space \( a_2, a_3, a_4 \) where different bifurcations occur. Figure 1 is a three-dimensional picture for case (a) together with the surface \( \Delta_3 = 0 \). Figure 2 is a cross section of Figure 1 where we fix \( a_4 \) at a particular value. We see the two regions corresponding to, respectively, the 4-point and 8-point bifurcation cases. The positive definiteness condition holds only inside of the parabola. Figure 3 is a three-dimensional picture for case (b) together with the surface \( \Delta_3 = 0 \). Figure 4 is a cross section of Figure 3 when we fix \( a_3 \) at a particular value. We see the positive definiteness condition holds only inside the shaded region, and it is in the region of \( \Delta_3 > 0 \), hence it only has the 8-point bifurcation case.
Figure 1. The three surfaces from the outside to the inside are
\[ \Delta = (a_4 + 3a_2^2)^3 - 27(a_2 a_4 - a_3^2 - a_2^3)^2 = 0, \]
\[ a_2 a_4 - a_3^2 = 0, \text{ and } \Delta_3 = 0. \] The inner part of each surface corresponds to \( \Delta > 0, a_2 a_4 > a_3^2 \) and \( \Delta_3 < 0. \)

Figure 2. Bifurcation region for case (a) with \( a_4 = 4. \)

We easily observe that 6-point bifurcation is not open, as we must impose an additional condition \( \Delta_3 = 0. \) On the other hand, the 4-point and 8-point cases both are open.
Finally, we illustrate the bifurcations of the two generic cases with the following two examples:
Example 3.4 (4-point case). Let \(a_2 = 1, a_3 = 1, a_4 = 4\), i.e.

\[
g(\mu, x, y) = \frac{1}{2} \mu (x^2 + y^2) - (x^4 + 6x^2y^2 + 4xy^3 + 4y^4).
\]

This will satisfy the condition of \(\Delta > 0\), \(a_2 > 0\), \(a_2a_4 > a_3^2\), which make \(-g(0, x, y)\) positive-definite. And the cubic equation (1) is \(-2w^3 - 3w^2 - w + 1 = 0\) with discriminant \(\Delta_3 < 0\), i.e. there is only one real root \(w\) of (1), and this forms a 4-point bifurcation case. From the conditions, we can see that this case is open.

Example 3.5 (8-point case). Let \(a_2 = 1, a_3 = 1, a_4 = 2\), i.e.

\[
g(\mu, x, y) = \frac{1}{2} \mu (x^2 + y^2) - (x^4 + 6x^2y^2 + 4xy^3 + 2y^4).
\]

This will satisfy the condition of \(\Delta > 0\), \(a_2 > 0\), \(a_2a_4 > a_3^2\), which make \(-g(0, x, y)\) positive-definite. And the cubic equation (1) is \(-2w^3 - 3w^2 + w + 1 = 0\) with discriminant \(\Delta_3 > 0\), i.e. there are three different real roots \(w\) of (1), and this forms an 8-point bifurcation case. From the condition, we can see that this case is also open.

The type of the bifurcating critical points will be alternating hyperbolic and elliptic surrounding the origin. As indicated in Figure 5 for the 4-point bifurcation case.

3.2.2. A highly degenerate case. It is a little surprising that, in generic situations, four or eight new fixed points are created by increasing the C-Z index of a known fixed point. One would normally expect to have cases with two new ones also. Similar to the example on \(S^2\) at the end of the first section of this paper, as one perturbs an integrable twist map, one expect to see one hyperbolic and one elliptic fixed point. However, it turns out that starting with an integrable twist map is not an open condition. It occurs in a highly degenerate case. We will give an example of such case in this section.
In this example, the quartic polynomial is equal to a perfect square, for example, \((x^2 + y^2)^2\). This correspond to the cases where \(a_3 = 0\) as in the last section. As a matter of fact, 2-point bifurcation is possible only when our quartic part is a perfect square (and positive definite as assumed). And since it is highly degenerate, the effect of \(\mu \mathcal{O}(|x| + |y|)^3\) in \(g(\mu, x, y)\) cannot be ignored. A typical 2-point bifurcation example is given in the following.

**Example 3.6 (2-point case).** Let

\[
g(\mu, x, y) = \frac{1}{2}\mu(x^2 + y^2) + \mu(x^2 + y^2)x - (x^2 + y^2)^2.
\]

Direct computations for the critical points of \(g(\mu, x, y)\) yield three critical points when \(\mu > 0\), which are \(y = 0, x = 0\) or \(\mu = 4x^2/(1 + 3x)\). And their bifurcation picture can be seen in Figure 6. The two bifurcating critical points are a local maximum and a saddle as indicated in the picture.

Again, the reason why we do not have this 2-point bifurcation generically when Conley-Zehnder index changes from 1 to 3 is that we started from Hamiltonian maps. When \(\mu = 0\), generically, the map will not come from an integrable Hamiltonian system. So when we transfer the map to a generating function, the generating function at \(\mu = 0\) will not be integrable, as we have seen in the former analysis for the quartic part of the generating function. And this leads to the open cases are the 4-point and 8-point cases for the bifurcation of fixed points of a one-parameter family of Hamiltonian maps.

On the other hand, if we consider a one-parameter Hamiltonian system \(\dot{Z} = JdH(\mu, Z)\) with \(Z = (x, y)\) and

\[
H(\mu, Z) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2
\]
Suppose it is an integrable system when $\mu = 0$, thus when $|\mu|$ is small, this system is a small perturbation of integrable Hamiltonian systems. Also we assume the origin is an isolated equilibrium with Conley-Zehnder index changes directly from 1 to 3 when $\mu$ passes zero. In this case, we believe the 2-point bifurcation case as indicated in Figure 6 is an open case.

### 3.3. Higher dimensional bifurcation

In this section, we study the bifurcations for higher dimensional hamiltonian diffeomorphisms. As one expects, generic co-dimension one bifurcations can all be reduced to two-dimensional cases. To see this, we first study how and when will the Conley-Zehnder index of a fixed point change.

Previously we have introduced the definition of Conley-Zehnder index, and we know that the Conley-Zehnder index is invariant under homotopy as long as the end point of the symplectic path stays in one of the connected components of $Sp(2n)^\pm$. In other words, the Conley-Zehnder index will change only when the end point passes through the component $Sp(2n)^0 = \{ A \in Sp(2n) | \det(A - I) = 0 \}$, i.e. with eigenvalues equal to one.

The eigenvalues of a symplectic matrix $A \in Sp(2n)$ has the form of quadruples: $\{ \lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1} \}$. And when $\lambda$ is real or on the unit circle, the strict quadruples reduce to a pair $\{ \lambda, \lambda^{-1} \}$. By the definition of the rotation function $\rho$ for the Conley-Zehnder index, we know that strict quadruples will not contribute to the rotation function. Since we only consider codimension-one bifurcation, the possible change of eigenvalues corresponding to the change of the Conley-Zehnder index increase by 2 should be that there is a pair of eigenvalues on the unit circle passes through one and still on the unit circle. Note that the case when a pair of eigenvalues on the unit circle passes through 1 to the real line will corresponds to Conley-Zehnder index increasing by one. And for the similar reason as below, the bifurcation of this case is essentially the same as its two-dimensional case which will be discussed in the last section.

The higher dimensional bifurcation should be essentially the same as the two-dimensional case. We will see this by the Lyapunov-Schmidt reduction.

Recall that the Lyapunov-Schmidt reduction says the following. Suppose $X, Y$ are Banach spaces, we may assume them to be $\mathbb{R}^n$ and $\mathbb{R}^m$ for simplicity. $G : X \to Y$ is a smooth function with $G(0) = 0$ and we want to solve $G(x) = 0$. We can write $G(x) = DG(0)x + \Gamma(x)$ where $DG(0)$ is the differential of $G$ at 0 and it is a continuous linear
operator, $\Gamma(x) = G(x) - DG(0)x$ is a smooth function with $D\Gamma(0) = 0$. Suppose $X_1 = \text{Ker}DG(0)$ and $Y_2 = \text{Range}DG(0)$. Let $X_2, Y_1$ be their complement spaces with $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Thus a point $x \in X$ can be uniquely written as $x = x_1 + x_2$ with $x_1 \in X_1, x_2 \in X_2$. Suppose that $Q_i : Y \to Y_i, i = 1, 2$ are projections of $Y$ to the two subspaces $Y_1, Y_2$. Thus we can write the equation $G(x) = 0$ in the following equivalent way:

\[
Q_1G(x_1 + x_2) = 0 \in Y_1
\]

\[
Q_2G(x_1 + x_2) = 0 \in Y_2
\]

Since $DG(0)|_{X_2} : X_2 \to Y_2$ is an isomorphism, the equation

\[
Q_2G(x_1 + x_2) = 0 \in Y_2
\]

can be solved in a neighborhood of the origin by

\[
x_2 = \varphi(x_1), \varphi(0) = 0
\]

where $\varphi$ is uniquely determined by the Implicit Function Theorem. If we plug this to the first equation, we get $Q_1G(x_1 + \varphi(x_1)) = 0 \in Y_1$. Thus, if we define $f : X_1 \to Y_1$ by $f(x_1) = Q_1G(x_1 + \varphi(x_1))$ and note that we have $f(0) = 0$, we have then reduced the question to find the local zero set of $f$, which is defined on a lower dimensional space. This is the main idea of Lyapunov-Schmidt reduction.

Now suppose we have a one-parameter Hamiltonian maps $F_\mu : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where $(\mathbb{R}^{2n}, \omega_0)$ is the standard $2n$-dimensional symplectic manifold. From the above analysis of eigenvalues of $DF_\mu(0)$ corresponding to the change of Conley-Zehnder index, we know there is one pair of eigenvalues on the unit circle passes through one when $\mu$ passes 0. Without loss, we may assume $DF_\mu(0) \in Sp(2n)$ has one pair of eigenvalues in the form of $e^{\pm i\mu}$, and the other eigenvalues are away from 1 when $\mu$ is small. Also the origin is always a fixed point of $F_\mu$ for all small $\mu$. We want to find the fixed points of $F_\mu$, that is to solve the equation of $G(\mu, x) = 0$, where $G(\mu, x) = x - F_\mu(x)$.

We may write $G(\mu, x) = x - DF_\mu(0)x + \Gamma(\mu, x)$, with $\Gamma$ satisfy the following conditions:

- $\Gamma(\mu, 0) = 0$, for all $\mu \in \mathbb{R}$;
- $\Gamma(\mu, x) = o(||x||)$ around the origin uniformly with respect to $\mu$ on bounded intervals;
- $D_x\Gamma(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$.

Since $G(\mu, x) : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, let $\bar{X}_1 = \mathbb{R} \times X_1$, where $X_1 = \text{Ker}(I - DF_0(0))$. Let $X_2$ be the complement of $X_1$, i.e. $X_1 \oplus X_2 = \mathbb{R}^{2n}$. 
\[ \tilde{X}_2 = \{0\} \times X_2. \] Thus, we have the function  \( \tilde{\varphi}(\tilde{x}_1) = \varphi(\mu, x_1) \) with values in  \( X_2 \), which is characterized by
\[ Q_2(I - DF_\mu(0))(x_1 + \varphi(\mu, x_1)) + Q_2 \Gamma(\mu, x_1 + \varphi(\mu, x_1)) = 0. \]
So, we have reduced the equation of \( G(\mu, x) = x - DF_\mu(0)x + \Gamma(\mu, x) = 0 \) to the equation:
\[ Q_1[(I - DF_\mu(0))(x_1 + \varphi(\mu, x_1)) + \Gamma(\mu, x_1 + \varphi(\mu, x_1))] = 0 \]
This is a function on a two-dimensional space. Therefore, it is clear that the bifurcation of fixed points of \( F_\mu \) is then reduced to the two-dimensional case we have discussed before.

4. Bifurcation when C-Z index changes from 1 to 2

In this section, we study the relatively easier case where the Conley-Zehnder index changes from 1 to 2. We use the same setting as in section 3. Locally \( F_\mu \) is an area-preserving map of a neighborhood of the origin in  \( \mathbb{R}^2 \):
\[ F_\mu : (x, y) \mapsto (X_\mu(x, y), Y_\mu(x, y)) \]
And the origin is an isolated fixed point for \( F_\mu \) in this neighborhood. Suppose the Conley-Zehnder index of the origin changes directly from 1 to 2 when \( \mu \) passes 0, then the eigenvalue of \( DF_\mu(0) \) should be on the unit circle and passes through 1 to the real line. As indicated in the following picture for the Krein-positive eigenvalue of \( DF_\mu(0) \):

This implies that the origin changes from elliptic to hyperbolic, and as in section 3 we can transfer the bifurcation of fixed points to the study of critical points of a proper generating function \( g_\mu \). The generating function will have the origin as a local maximum or minimum when \( \mu < 0 \) and a saddle when \( \mu > 0 \). Without loss, we can assume it is a maximum when \( \mu < 0 \). Again by Conley index theory for isolated invariant set, there must be at least two critical points bifurcate from the origin when \( \mu \) passes 0 provided the critical points are all non-degenerate.
Lemma 4.1. Suppose $g(\mu, z)$ is a one-parameter smooth function on $\mathbb{R}^2$ such that the origin is an isolated critical point for all $\mu \in [-1, 1]$. And it is a local maximum for $\mu \leq 0$, a saddle for $\mu > 0$. Then there are at least two critical points bifurcate from the origin when the critical points of $g$ are nondegenerate.

Proof. The proof is almost the same as in Lemma 3.2. As we collapse the exit set for $N_\mu = N_0 \setminus D_\mu$, we will get that the index of its invariant set is $(S^1 \vee S^1, \cdot)$. This index has betti number $\beta_1 = 2$. Thus, when the critical points are nondegenerate, the Morse theorem implies that there are at least two more rest points. □

For a more detailed analysis of the bifurcation in this case, without loss of generality, we may assume that the quadratic term of $g(\mu, x, y)$ is $\frac{1}{2}(\mu x^2 - y^2)$. And we also assume the origin is a local maximum when $\mu = 0$. That is, our function is $g(\mu, x, y) = \frac{1}{2}(\mu x^2 - y^2) + \text{h.o.t.}$, and we require that $-g(0, x, y)$ to be positive definite in a small neighborhood of the origin. The term $x^3$ is not positive definite if one sets $y = 0$, so it won’t appear. Also it not difficult to see that the term $x^2y$ does not show up either. The first term in $x$ alone is $x^4$. Therefore,

\[ g(\mu, x, y) = \frac{1}{2}(\mu x^2 - y^2) + (c_1 xy^2 + c_2 y^3) + (a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4) + \mu(b_0 x^3 + b_1 x^2 y + b_2 xy^2 + b_3 y^3) + \text{h.o.t.} \]

We only consider the nondegenerate case where the coefficient of $x^4$ is nonzero. Since $-y^2$ dominate the $y$-direction, the function $-g(0, x, y)$ is positive definite in a small neighborhood of the origin only if $a_0 < 0$. To find the critical point of $g(\mu, x, y)$, we focus on the equation of $g_x = g_y = 0$.

\[ g_x(\mu, x, y) = \mu x + c_1 y^2 + \mu(O(|x| + |y|)^2) + 4a_0 x^3 + \cdots \]
\[ g_y(\mu, x, y) = -y + 2c_1 xy + 3c_2 y^2 + \mu(O(|x| + |y|)^2) + \cdots \]

Note that the $O(|x| + |y|)^k$ denotes $k$-th degree homogeneous polynomials in terms of $x$ and $y$. And we will see that only the higher order term $4a_0 x^3$ in $g_x$ will play an important role in the bifurcation.

Firstly, we can reduce the bifurcation to a one-dimensional case. The equation $g_y = 0$ tells us that $g_{yy}(0, 0, 0) = -1 \neq 0$. By Implicit Function Theorem, there exists a unique smooth function $\varphi : (\mu, x) \mapsto y$ in a small neighborhood of $(0, 0, 0)$ such that

\[ g_y(\mu, x, \varphi(\mu, x)) = 0 \]
Moreover, \( \varphi(0,0) = 0, \varphi_\mu(0,0) = 0, \varphi_x(0,0) = 0 \). Thus our question can be reduced to the equation of \( G(\mu, x) = 0 \), where

\[
G(\mu, x) = g_x(\mu, x, \varphi(\mu, x))
\]

Consider the Taylor expansion of \( \varphi(\mu, x) \) near \( (0, 0) \), since \( \varphi(0, 0) = 0, \varphi_\mu(0,0) = 0, \varphi_x(0,0) = 0 \), it should start with the second order term:

\[
\varphi(\mu, x) = \frac{1}{2}(\varphi_{\mu\mu}(0,0)\mu^2 + \varphi_{\mu x}(0,0)\mu x + \varphi_{xx}(0,0)x^2) + \text{h.o.t.}
\]

Plug this into \( G(\mu, x) \) we get:

\[
G(\mu, x) = \mu x + b\mu x^2 + 4a_0 x^3 + \mathcal{O}((|x| + |\mu|)^4) + \text{h.o.t.}
\]

It is clear that the dominant part for the equation \( G(\mu, x) = 0 \) is \( \mu x + b\mu x^2 + 4a_0 x^3 \), where \( b \) is a real constant, and \( a_0 \) is as in \( g_x \) above with \( a_0 < 0 \). Moreover, \( G(\mu, x) = 0 \) implies that \( \mu = \mathcal{O}(x^2) \) near the origin, thus the term \( b\mu x^2 \) is a higher order term, which will not affect this bifurcation. Therefore, our bifurcation has been reduced to a one-dimensional pitchfork bifurcation with normal form \( G(\mu, x) = \mu x - x^3 \).

For our original two-dimensional bifurcation, the typical bifurcation is the following example.

**Example 4.2** (Maximum to saddle bifurcation). Let

\[
g(\mu, x, y) = \frac{1}{2}(\mu x^2 - y^2) - \frac{1}{4}x^4.
\]

Direct computations for the critical points of \( g(\mu, x, y) \) yield three critical points when \( \mu > 0 \), which are \((0, 0)\) and \((\pm\sqrt{\mu}, 0)\). And their bifurcation picture can be seen in Figure 7. The two bifurcating critical points are both local maximum as indicated in the picture.

**Conclusion.** In this paper, we have studied the bifurcation of fixed points of a one-parameter family of Hamiltonian maps when the Conley-Zehnder index of a given fixed point increases. There are two
possible cases: the index increases by one or two. We have given detailed analysis for each of these cases. The more interesting case is when the index increases by 2, i.e. from 1 to 3. This case corresponds to the generating function has a critical point changing from a local maximum to a local minimum. Under generic (open) conditions, there are four or eight critical points bifurcating from the original one, and their stability type is alternatingly hyperbolic and elliptic surrounding the original critical point. A special non-generic case where two critical points bifurcate from the original one is also given. The relative easier case is when the Conley-Zehnder index increases by 1, i.e. from 1 to 2. This case corresponds to where the generating function has a critical point changing from a local maximum (or a local minimum) to a saddle. The generic case for this bifurcation is that there are two critical points bifurcating from the original one and both of them are local maximum (or local minimum). Note that K.R.Meyer [10] has analysed the bifurcation of fixed points of another generic one-parameter family of symplectic diffeomorphisms on a two manifold, namely the saddle-center bifurcation. In our work, we have studied the bifurcation of fixed points when the Conley-Zehnder index of a given fixed point changes.

REFERENCES


E-mail address: dengyx@math.northwestern.edu, xia@math.northwestern.edu