Rational Solutions to the Toda Lattice Hierarchy and Irreducible Representations of the Virasoro Algebra

Ivan Dimitrov *

Differential Equations,
Faculty of Mathematics and Informatics,
Sofia University, 5 J. Bourchier, 1126 Sofia, Bulgaria

Abstract

We build higest weight representations of the Virasoro algebra with highest weight vectors tau-functions of Toda deformations of Laguerre polynomials. Then we describe all rationals solutions of the Toda system and prove that the corresponding tau-functions are highest weight vectors of irreducible degenerate representations of the Virasoro algebra with central charge $c = 1$.

Key words (1991 MSC): Primary 35Q53; Secondary 81R10, 33C45.

Introduction

The present work is inspired by recent development in the mathematical aspects of 2d-quantum gravity. It was found (see [3] and [4]) that the partition function of descrete 2d-quantum gravity on one hand is a tau-function of Toda deformations of Hermite polynomials and on the other hand generates a highest weight representation of the Virasoro algebra Vir with highest weight $c = 1$ and $h = 0$. For a detailed presentation of above mentioned questions see also [7].

In [5] Haine and Horozov considered the partition functions of the matrix models

$$Z_n^\alpha(t) = \int e^{\alpha \log trM - tr(M+V(M))} dM \ , \ \alpha > \ -1,$$

where M is a positive definite Hermitian matrix and $V(x) = \sum t_k x^k$. They showed that these functions are tau-functions of Toda deformations of Laguerre polynomials $\{p_n^\alpha(x)\}_{n=0}^\infty$ with $\alpha > -1$ (for precise definitions see below). Again these partition functions provide highest weight representations of the Virasoro algebra but now the energy h can be any non-negative number. It is a natural question if the tau-functions of the other Laguerre polynomials ($\alpha \leq -1$) provide representations of Vir. The present work answers this question.

The Laguerre polynomials are defined via [2]:

$$p_n^\alpha(x) = (-1)^n e^x x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}). \tag{0.1}$$

*Grant MM1/91 NFSR is gratefully acknowledged

When $\alpha > -1$ they are orthogonal with respect to the measure $d\sigma_\alpha = x^\alpha e^{-x} dx$ on the interval $[0, \infty)$ (see also [1]). (Actually in [5] this is used as a definition). Unfortunately, in the case $\alpha \leq -1$ there is no positive definite integral measure $d\sigma_\alpha$ such that $\{p_n^\alpha\}_{n=0}^\infty$ are orthogonal basis with respect to it. In fact there exists non-degenerate bi-linear form $< \circ, \circ >_\alpha$ in the space of all polynomials $\mathbb{C}[x]$ such that $\{p_n^\alpha\}_{n=0}^\infty$ are orthogonal basis with respect to $< \circ, \circ >_\alpha$. This observation allows us to state that the Toda deformations of **all** Laguerre polynomials provide highest weight representations of Vir with $(c, h) = (1, \frac{\alpha^2}{4})$.

A natural question arises: what is the structure of these representations. They turn out to be quite closely connected with the structure of $< \circ, \circ >_\alpha$.

There are two different cases: $\alpha \leq -1$ non-integer and α negative integer. When α is non-integer the form $< \circ, \circ >_\alpha$ is defined via the integral measure considered in [5]. However, when $\alpha = -m$ is negative integer, $\mathbb{C}[x]$ splits into direct sum of the m-dimensional space *span* $\{1, x, \ldots, x^{m-1}\}$ and the infinite dimensional $x^m\mathbb{C}[x]$ which are mutually orthogonal with respect to $< \circ, \circ >_\alpha$. The tau-functions corresponding to the space *span* $\{1, x, \ldots, x^{m-1}\}$ are exactly the Schur polynomials which are described in [6] to be highest weight vectors of Vir. The tau-functions produced from $x^m\mathbb{C}[x]$ coincide with the tau-functions obtained in [5] with $\alpha = m$.

Notice that the above mentioned Schur polynomials provide rational solutions to the Toda system. Moreover, we prove that **all** rational solutions to the Toda lattice hierarchy are given by these polynomials. As it is known [6] the representations with $(c, h) = (1, \frac{\alpha^2}{4})$ when α is non-integer are irredusible and isomorphic to the Verma representation $V(1, \frac{\alpha^2}{4})$. The case of integer α is more complicated.The representations obtained by Haine and Horozov are almost all redusible (apart from these with $Z_0^\alpha(t) \equiv 1$). As we noted above the tau-functions in the case of negative integer α are of two kinds: In the first case Schur polynomials provide irredusible representations of Vir (see [6]). In the second one — these studied by Haine and Horozov — provide redusible representations of Vir.

Thus we explicitely state the conection between the rational solutions to the Toda system and irredusible representations of the Virasoro algebra (see Theorem 2.2 and Theorem 3.1).

Acknowlegements: I would like to thank E. Horozov for introducing me into the theme as well as L. Haine for helpful discussions conserning the Toda lattice hierarchy.

1. Toda hierarchy and its tau-functions

In this section we briefly recall the construction of tau-functions of Toda deformations of any orthogonal polynomials. We strictly follow [5] ommiting some of details.

Consider a measure $d\sigma_0(x)$ on an interval (a, b) . Define the deformed measure

$$d\sigma_t(x) = e^{-V(x)} d\sigma_0(x), \tag{1.1}$$

where $V(x) = \sum_{k=1}^\infty t_k x^k$.

Let $p_n(x; t) = x^n + \cdots$ $(n \geq 0)$ be the system of monic polynomials orthogonal with respect to (1.1). Then $\{p_n\}_{n=0}^\infty$ satisfies

$$\int_a^b p_m(x; t) p_n(x; t) d\sigma_t(x) = h_m(t) \delta_{mn}. \tag{1.2}$$

As with any system of orthogonal polynomials, the operator of multiplication by x in the basis $\{p_n(x;t)\}$ is represented by an infinite tridiagonal matrix $Q(t) = (Q_{mn}(t))_{m,n\geq 0}$:

$$xp_n(x;t) = p_{n+1}(x;t) + Q_{nn}(t)p_n(x;t) + Q_{n-1,n}(t)p_{n-1}(x;t). \tag{1.3}$$

Q provides a solution to the infinite dimensional Toda lattice hierarchy:

$$\frac{\partial Q}{\partial t_j} = [Q, (Q^j)_+], \tag{1.4}$$

where $(Q^j)_+$ is the strictly upper part of Q^j. Define

$$\tau_n(t) = h_0(t)\cdots h_{n-1}(t) \tag{1.5}$$

and by convention $\tau_0(t) = 1$. It is well-known that the ellements of $Q(t)$ are given by

$$Q_{n-1,n}(t) = \frac{\tau_{n-1}(t)\tau_{n+1}(t)}{\tau_n^2(t)}, \quad Q_{nn}(t) = \frac{\partial}{\partial t_1}\log\frac{\tau_n(t)}{\tau_{n+1}(t)}. \tag{1.6}$$

Consider the *wave matrix* $W(t) = (W_{ij}(t))_{i,j\geq 0}$ which is given by $p_k(t) = \sum_{k=1}^{\infty} w_{ik}x^k,\ k = 0, 1, \ldots$ [1]. The following formulas can be found in [5].

Proposition 1.1 *For each $n \geq 0$, we have (up to a constant):*

$$\begin{aligned}
\tau_n(t) &= <\exp(\sum_{k=1}^{\infty} t_k\Lambda^{-k})W(0)\circ\Phi_n, \Phi_n> &(1.7)\\
&= <\exp(-\sum_{k=1}^{\infty} t_k\Lambda^{k})W^*(0)\circ\Phi_n^*, \Phi_n^*> &(1.8)\\
&= <\exp(-\sum_{k=1}^{\infty} t_k\Lambda^{k})M(0)\circ\Phi_n^*, \Phi_n^*> &(1.9)\\
&= D_n(t), &(1.10)
\end{aligned}$$

where Φ_n and Φ_n^* are the vacuum and the dual vacuum vectors in the in the Fock space of free fermions and in the dual Fock space respectively and $\circ$ in (1.7), (1.8) *and* (1.9) *is to be understood as a group action on the vacuum vector Φ_n and Φ_n^*; $W^*(0) = (W(0)^T)^{-1}$; $M(t)$ is defined with*

$$M_{ij} = \left\{\begin{array}{cc} \delta_{ij} & i<0 \text{ or } j<0 \\ \mu_{i+j}(t) & i\geq 0 \text{ and } j\geq 0, \end{array}\right. \tag{1.11}$$

where

$$\mu_k(t) = \int_a^b x^k d\sigma_t(x), \tag{1.12}$$

and $D_n(t) = \det(\mu_{i+j}(t))_{0\leq i,j\leq n-1}$.

[1]In the sequel when the dependence on t of $Q(t)$, $L(t)$, $p = n(x;t)$, etc. is not emphasized we shall just write Q, L, $p_n(x)$, etc.

Remark. Formulas (1.7) – (1.9) represent $\tau_n(t)$ as determinants — formula (1.7) as infinite and (1.8) and (1.9) — as finite determinants.

In the rest of this section we specialize on the case of Laguerre polynomials; defined by (0.1). In the case $\alpha > -1$ they are orthogonal with respect to the measure

$$d\sigma_0^\alpha(x) = x^\alpha e^{-x} dx, x \in (0, \infty). \tag{1.13}$$

As it is proved in [5] the flag $\{W_n^\alpha\} = W_0^\alpha \supset W_1^\alpha \supset \cdots$ defined with $W_n^\alpha = span\{p_n^\alpha(x), p_{n+1}^\alpha(x), \ldots\}$ is invariant under the action of the operator $A = -x\dfrac{d}{dx} + x$, i.e.

$$Ap_n = p_{n+1} + b_n p_n. \tag{1.14}$$

Also the flags $\{W_n^\alpha\}$ with $\alpha > -1$ are shown to be the unique Toda flags (i.e. flags spaned by polynomials with tri-term relation (1.3)) which are invariant under the action of A with the additional condition $b_0 > 0$. It is possible to avoid this condition. We give below a full describtion of the Toda flags invariant under A:

Proposition 1.2 *Every Toda flag $\{W_n\}$, $W_n = \mathrm{span}\{q_n, q_{n+1}, \ldots\}$, invariant under A is either*
(i) spanned by Laguerre polynomials $\{p_n^\alpha\}$ for some α, or
(ii) $b_0 = -m$ is a negative integer; $q_n = p_n^{-m}$ for $n = 0, 1, \ldots, m$; $q_n = x^m \widetilde{q_{n-m}}$ for $n \geq m$ and the flag $\{\widetilde{W_n}\}$ spanned by $\{\widetilde{q_n}\}$ is invariant under A as well.

The proof is based on a careful modification of the similar proof of the above mentioned theorem in the case $b_0 > 0$ and that is why we omit it here.

What we are going to do is to perform the whole construction of $\tau_n(t)$ as a tau-function of a Toda deformation of the Laguerre polynomials in the case $\alpha \leq -1$. Although they are not orthogonal with respect to any measure (as it will be seen below) it is possible to adapt the construction from [5] to the case of interest. Proposition 1.2 and the formulas (1.7) show that the flags described in the Proposition 1.2 (*ii*) do not give new tau-functions. In other words, the functions studied in [5] and those we are going to obtain (i.e. tau-functions of Toda deformations of the Laguerre polynomials) are the unique tau-functions of Toda deformations of flags invariant under the action of A.

2. Laguerre polynomials. Case $\alpha \leq -1$

In this section we are going to introduce a bi-linear symetric form $< \circ, \circ >_\alpha$ on the space $\mathbb{C}[x]$ such that $\{p_n^\alpha(x)\}, \alpha \leq -1$ is orthogonal basis in $\mathbb{C}[x]$ with respect to $< \circ, \circ >_\alpha$. Having this form we can define the function $\tau_n(t)$ as a tau-function of the Toda deformation of Laguerre polinomials. This definition allows us to use the techniques developed in [5] to prove that $\tau_n(t)$ provides highest weight representation of Vir.

Now we define the form $< \circ, \circ >_\alpha$. There are two principally different cases: integer α and non-integer α. Let $\alpha = -m$ be a negative integer. Then

$$< p(x;t), q(x;t) >_\alpha = \frac{(-1)^{\frac{m(m-1)}{2}}}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} (pqe^{-x-V(x)}) \,|_{x=0} + \int_0^\infty x^m \tilde{p}\tilde{q} e^{-x-V(x)}) dx \tag{2.1}$$

where the wave is the operator which cuts off all powers up to $m-1$, i.e.

$$\tilde{p} = a_m x^m + a_{m+1}x^{m+1} + \ldots, \quad \text{if} p = a_0 + a_1 x + \ldots.$$

Now let $\alpha < -1$ be non-integer and $-m-1 < \alpha < -m$. Define

$$< p(x;t), q(x;t) >_\alpha = \frac{(-1)^m}{(\alpha+1)\ldots(\alpha+m)} \int_0^\infty x^{\alpha+m} \frac{d^m}{dx^m}(pqe^{-x-V(x)})dx. \tag{2.2}$$

Theorem 2.1 *Let $\alpha \leq -1$. Then the bi-linear form $< \circ, \circ >_\alpha$ is non-degenerate and the Laguerre polynomials* (0.1) *are the orthogonal basis in the space of all complex polynomials.*

Remark. We call the system of monic polynomials $\{q_n\}_{n=0}^\infty$, where $\deg q_n = n$, orthogonal basis with respect to the form $< \circ, \circ >$ if q_n is the unique monic polynomial of degree n, such that $< q_n, q >= 0$ for every polynomial q with $\deg q < n$.

Proof. The case of non-integer α is easy to treat . Suppose $\alpha > -1$ and consider the scalar product assosiated with the measure (1.1):

$$\begin{aligned} < P, Q >_\alpha &= \int_0^\infty x^\alpha P(x)Q(x)e^{-x-V(x)}dx \\ &= -\frac{1}{\alpha+1}\int_0^\infty x^{\alpha+1}\frac{d}{dx}(P(x)Q(x)e^{-x-V(x)})dx = \ldots \\ &= -\frac{(-1)^m}{(\alpha+1)\ldots(\alpha+m)}\int_0^\infty x^{\alpha+m}\frac{d^m}{dx^m}(P(x)Q(x)e^{-x-V(x)})dx. \end{aligned} \tag{2.3}$$

Since $\{p_n^\alpha\}_{n=0}^\infty$, $\alpha > -1$ is orthogonal basis with respect to (2.3) and the dependence on α of p_n^α is polynomial it follows that $\{p_n^\alpha\}_{n=0}^\infty$, $\alpha < -1$ and α — non-integer, is orthogonal basis with respect to (2.2).

The case of negative integer α is more complicated. Let $\alpha = -m$ and $\{q_n(t)\}_{n=0}^\infty$ be the orthogonal basis with respect to (2.1) (it will be seen below that $< \circ, \circ >_\alpha$ is non-degenerate and hence such a basis exists). From the definition (2.1) it is clear the subspace *span*$\{1, x, \ldots, x^{m-1}\}$ and $x^m\mathbb{C}[x]$ of $\mathbb{C}[x]$ are mutually arthogonal and hence q_n divides by x^m and

$$q_n = x^m p_{n-m}^{-\alpha}(x;t), \quad \text{if} \quad n \geq m. \tag{2.4}$$

On the other hand it is a simple computation to check that $p_n^\alpha(x;t) = x^m p_{n-m}^{-\alpha}(x;t)$.

Let us concentrate on $n < m$. By the end of this section whenever we omit the argument t it means that it is taken at $t = 0$. It is easy to see (following the well-known procedure) that $\{q_n\}_{n=0}^{m-1}$ provides a solution to the finite Toda lattice hierarchy:

$$xq_n(x;t) = q_{n+1}(x;t) + Q_{n,n}q_n(x;t) + Q_{n-1,n}(t)q_{n-1}(x;t). \tag{2.5}$$

Define $\mu_n^\alpha(t)$ with

$$\mu_n^\alpha(t) =< x^n, 1 >_\alpha = (-1)^{\frac{m(m-1)}{2}} S_{m-1-n}(-1-t_1, -t_2, \ldots), \tag{2.6}$$

where $S_k(t)$ are the standard Schur polynomials and $D_n(t) = \det(\mu_{i+j}^\alpha(t))_{0\leq i,j\leq n-1}, D_0(t) = 1$. The non-degeneracy of $< \circ, \circ >_\alpha$ is equivalent to the condition $D_n(t) \neq 0$ for $n = 0, 1, \ldots, m-$

1. In the sequel we shall compute explicitely $D_n(0)$ and see that $< \circ, \circ >_\alpha$ is non-degenerate. Suppose in a moment it is done. Then [1]

$$q_n(x;t) = \frac{1}{D_n(t)} \begin{vmatrix} \mu_0(t) & \dots & \mu_n(t) \\ \vdots & & \vdots \\ \mu_{n-1}(t) & \dots & \mu_{2n-1}(t) \\ 1 & \dots & x^n \end{vmatrix} \tag{2.7}$$

and $h_n(t) =< q_n(x;t), q_n(x;t) >_\alpha = \dfrac{D_{n+1}(t)}{D_n(t)}$ for $n < m$. Now it is a standard procedure to find

$$\begin{aligned} Q_{n,n+1}(t) &= \frac{h_{n+1}(t)}{h_n(t)} &&, \text{ if } \ n < m-1 \\ Q_{m-1,m}(t) &= 0 && \\ Q_{n,n}(t) &= \frac{\partial}{\partial t_1} \log \frac{1}{h_n(t)} &&, \text{ if } \ n < m-1. \end{aligned} \tag{2.8}$$

What we need to end the proof is to compute the coefficients from (2.8) at $t = 0$ and compare them with these representing the multiplication by x in the basis $\{p_n^\alpha\}_{n=0}^{m-1}$:

$$xp_n^\alpha(x) = p_{n+1}^\alpha(x) + (2n+1+\alpha)p_n^\alpha(x) + n(n+\alpha)p_{n-1}^\alpha(x). \tag{2.9}$$

From (2.6) we find

$$\mu_n^\alpha = \begin{cases} \dfrac{(-1)^{\frac{m(m-1)}{2}+m-1-n}}{(m-1-n)!} &, \text{ if } \ n < m \\ \\ 0 &, \text{ if } \ n \geq m. \end{cases} \tag{2.10}$$

Then $D_n = (-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{n(n-1)}{2}} \dfrac{1!2!\dots(n-1)!}{(m-1)!\dots(m-n)!}$, if $n \leq m$.

Hence $h_n = \dfrac{D_{n+1}}{D_n} = (-1)^n \dfrac{n!}{(m-n-1)!}$ and

$$Q_{n,n+1} = \frac{h_{n+1}}{h_n} = -(n+1)(m-n-1), \text{ if } \ n < m-1 \tag{2.11}$$

and $Q_{m-1,m} = 0$. To compute $Q_{n,n}$ we use

$$D_n(t) = (-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{n(n-1)}{2}} S_{\underbrace{m-n,\dots,m-n}_{n \text{ times}}}(-1-t_1, -t_2, \dots), \text{ if } \ n \leq m-1, \tag{2.12}$$

with the standard Schur polynomials $S_{j_1,\dots,j_s}(t)$. Then

$$\frac{\partial}{\partial t_1} D_n(t)\,|_{t=0} = (-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{n(n-1)}{2}} S_{\underbrace{m-n,\dots,m-n-1}_{n \text{ times}}}(-1-t_1, -t_2, \dots), \text{ if } \ n \leq m-1$$

and $\frac{\partial}{\partial t_1} D_m(t)\,|_{t=0} = 0$. Finaly

$$
\begin{aligned}
Q_{n,n} &= -\frac{\partial}{\partial t_1}\log\frac{D_{n+1}}{D_n} = \frac{1}{D_n}\log D_n - \frac{1}{D_{n+1}}\log D_{n+1} \\
&= n(m-n)-(n+1)(n-m-1) = (2n+1-m). \qquad (2.13)
\end{aligned}
$$

Comparing (2.11) and (2.13) with (2.9) we conclude that $q_n = p_n^\alpha$ for $n < m$. This ends the proof of the theorem.

Remark. There is no non-degenerate integral measure $d\mu$ such that $\{p_n^{-m}\}_{n=0}^\infty$ be orthogonal with respect to it. Indeed if it exists then $p_m^{-m} = x^m$ and $x^{2m} \in span\{p_m^{-m},\ p_{m+1}^{-m},\ldots\}$ and hence

$$
0 \neq \int p_m^{-m} p_m^{-m} d\mu = \int \sum_{k=m}^{2m} c_k p_k^{-m} d\mu = \sum_{k=m}^{2m} c_k \int 1.p_k^{-m} d\mu = 0.
$$

Now we are in position to state the following

Theorem 2.2 *Let $\tau_N^\alpha(t)$ be the functions*

$$
\tau_N^\alpha(t) = h_0(t)\ldots h_{N-1}(t),\ \tau_0^\alpha(t) = 1. \qquad (2.14)
$$

Then

(i)
$$
(L_j^{(\alpha,n)} + \frac{\partial}{\partial t_{j+1}})\tau_n^\alpha(t) = \frac{\alpha^2}{4}\delta_{0j}\tau_n^\alpha(t)\ ;\ \ j = 0,1,\ldots \qquad (2.15)
$$

where

$$
\begin{aligned}
L_0^{(\alpha,n)} &= \frac{(\mu_n^\alpha)^2}{2h} + \sum_{k=1}^\infty k t_k \frac{\partial}{\partial t_k} \\
L_j^{(\alpha,n)} &= \frac{1}{2h}\sum_{k=1}^{j-1}\frac{\partial^2}{\partial t_k \partial t_{j-k}} + \frac{\mu_n^\alpha}{h}\frac{\partial}{\partial t_j} + \sum_{k=1}^\infty k t_k \frac{\partial}{\partial t_{k+j}} \quad (j \geq 1)
\end{aligned} \qquad (2.16)
$$

with $h = \frac{1}{2}$ and $\mu_n^\alpha = -(n + \frac{\alpha}{2})$.
(ii) Let $\alpha = -m$ be negative integer. Then

$$
\tau_n^\alpha(t) = \begin{cases} (-1)^{\frac{m(m-1)}{2} + \frac{n(n-1)}{2}} S\underbrace{m-n,\ldots,m-n}_{n \text{ times}}(-1-t_1,-t_2,\ldots), & n < m \\ \tau_{-m+n}^m(t) & n \geq m. \end{cases}
$$

The proof of statement (ii) is contained in the proof of Theorem 2.1 while the proof of (i) is simply a repetition of a similar proof from [5] when the integral scalar product is replaced by the bi-linear form (2.1) or (2.2).
Remark. Put

$$
L_{-j}^{(\alpha,n)} = \frac{h}{2}\sum_{k=1}^{j-1} k(j-k)t_k t_{j-k} + \mu_n^\alpha j t_j + \sum_{k=j+1}^\infty k t_k \frac{\partial}{\partial t_{k-j}},\ j = 1,2,\ldots \qquad (2.17)
$$

Then the operators $L_j^{\alpha,n}$ from (2.16) and (2.17) satisfy the commutation relations of the Virasoro algebra with central charge $c = 1$. Notice that by the shift $t \to t + 1$ the operators L_j $(j \geq 0)$ transform into $\widetilde{L_j} = L_j^{(\alpha,n)} + \dfrac{\partial}{\partial t_{j+1}}$. Thus Theorem 2.2 simply claims that τ_N^α provides highest weight representation of the Virasoro algebra and $\tau_n^\alpha(t)$ with α negative integer and $n < -\alpha$ are the well-known (see [6]) Schur polynomials which are highest weight vectors.

Before finishing the section we remind two result about the representations from Theorem 2.2 (see [6]): The representations with $(c, h) = (1, \frac{\alpha^2}{4})$ being highest weight representations are irreducible for any non-integer α. In the case of integer α the representations from Theorem 2.2 are irreducible when $n = 0$ or $\alpha = -m$ is negative integer and $n \leq m$. In the next section we connect this fact with a theorem of uniqueness of rational solutions to the Toda hierarchy equations.

3. Rational solutions to the Toda lattice hierarcly

Consider the infinite Toda system (1.4) where

$$Q = \begin{pmatrix} Q_{00} & Q_{01} & 0 & 0 & \dots \\ 1 & Q_{11} & Q_{12} & 0 & \dots \\ 0 & 1 & Q_{22} & Q_{23} & \dots \\ 0 & 0 & 1 & Q_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} B_1 & A_1 & & & 0 \\ 1 & B_2 & A_2 & & \\ & 1 & B_3 & A_3 & \\ & & 1 & B_4 & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix} \tag{3.1}$$

It is well-known that every set of monic orthogonal polynomials provides a solution to (1.4). Here we briefly recall this constuction: Let $d\sigma_0(x)$ be some measure on an interval (a, b). Define the measure $d\sigma_t(x) = \exp\left(-\sum_{k=1}^\infty t_k x^k\right) d\sigma_0(x)$. Associated with it, consider the time dependent system of monic orthogonal polynomials (in x) $p_n(x; t)$ with respect to $d\sigma_t(x)$, satisfying

$$\int_a^b p_m(x; t) p_n(x; t) d\sigma_t(x) = h_m(t)\delta_{mn}.$$

As with any system of orthogonal polynomials, multiplication by x in the basis $\{p_n(x; t)\}_{n=0}^\infty$ is represented by an infinite tridiagonal matrix $Q(t) = (Q_{mn}(t))_{m,n\geq 0}$:

$$xp_n(x; t) = p_{n+1}(x; t) + Q_{nn}(t)p_n(x; t) + Q_{n-1,n}(t)p_{n-1}(x; t).$$

Then (see [7]) $Q(t)$ provides a solution to (1.4).

We need to reverse the above construction for our purposes. Let (3.1) be a solution to (1.4). We define a set of time-dependent polynomials of x with

$$\begin{aligned} p_0(x; t) &= 1 \\ xp_n(x; t) &= p_{n+1}(x; t) + B_{n+1}(t)p_n(x; t) + A_n(t)p_{n-1}(x; t). \end{aligned} \tag{3.2}$$

Put

$$h_n(t) = A_1(t) \dots A_n(t) h_0(t), \tag{3.3}$$

where $h_0(t)$ is defined by

$$h_0(0) = 1, \quad \frac{\partial h_0}{\partial t_i} = -(Q^i)_{00} h_0. \tag{3.4}$$

Lemma 3.1 *The function $h_0(t)$ exists.*

Proof. The existence of h_0 is equivalent to the existence of $\log f(t)$ such that $(f(t) = -h_0(t))$:

$$\frac{\partial}{\partial t_i} \log f(t) = (Q^i)_{00}.$$

To prove this it is sufficient to see that

$$\frac{\partial}{\partial t_j}(Q^i)_{00} = \frac{\partial}{\partial t_i}(Q^j)_{00}$$

if $Q(t)$ is a solution to (1.4). We have

$$\frac{\partial Q^j}{\partial t_i} = [Q^j, Q^i_+]$$

and hence

$$\frac{\partial Q^j}{\partial t_i} = [Q^j, Q^i_+] = [Q^j_+ + Q^j_-, Q^i_+] = [Q^j_+, Q^i_+] + [Q^j_-, Q^i_+] \text{ and}$$

$$\frac{\partial Q^i}{\partial t_j} = [Q^i, Q^j_+] = [Q^i, Q^j - Q^j_-] = [Q^j_-, Q^i] = [Q^j_-, Q^i_+ + Q^i_-] = [Q^j_-, Q^i_+] + [Q^j_-, Q^i_-].$$

The observation that $[Q^j_+, Q^i_+]$ is strictly upper matrix and $[Q^j_-, Q^i_-]$ — strictly lower matrix completes the proof of the lemma.

Here we state a lemma which will be used below:

Lemma 3.2

$$\frac{\partial h_n}{\partial t_i} = -(Q^i)_{nn} h_n. \tag{3.5}$$

Proof. For $n = 0$ the statement (3.5) is just the definition (3.4). Let $\frac{\partial h_n}{\partial t_i} = -(Q^i)_{nn} h_n$. Then

$$\frac{\partial h_{n+1}}{\partial t_i} = \frac{\partial (h_n A_{n+1})}{\partial t_i} = \frac{\partial h_n}{\partial t_i} A_{n+1} + h_n \frac{\partial A_{n+1}}{\partial t_i} = -(Q^i)_{nn} h_n A_n + [Q, Q^i_+]_{n-1,n} h_n.$$

Now (3.5) is equivalent to

$$[Q, Q^i_+]_{n,n+1} = (Q^i_{nn} - Q^i_{n+1,n+1}) A_{n+1}.$$

The last equality is a property of tridiagonal matrices of the type (3.1).

Suppose $< \circ, \circ >$ is a bi-linear symetric form on the space of complex polynomials $\mathbb{C}[x]$. The condition

$$< p_n(x;t), p_m(x;t) >= \delta_{nm} h_n(t) \tag{3.6}$$

uniquelly defines the quantities $\mu_{ij} =< x^i, x^j >$. It follows from (3.2), (3.3) and (3.6) that $< xp_n, p_m >=< p_n, xp_m >$. The polynomials $\{p_n\}_{n=0}^{\infty}$ form a basis in $\mathbb{C}[x]$ and thus

$$< xP, Q >=< P, xQ >$$

for arbitrary polynomials P and Q. In particular

$$\mu_{i+1,j} =< x.x^i, x^j >=< x^i, x.x^j >= \mu_{i,j+1}$$

which means that $\mu_{ij} = \mu_{i+j,0}$. Further we shall use $\mu_n = \mu_{n0}$ instead of μ_{ij}. We have

$$\mu_n =< 1, x^n >=< x^n \circ p_0, p_0 >$$

where $x^n \circ p_0$ is the action of the $n-$th degree of the operator of multiplication by x on the polynomial p_0. This means

$$\mu_n(t) = (Q^n)_{00} h_0(t). \tag{3.7}$$

Put $\dfrac{\partial p_k}{\partial t_i} = \sum_{j \le k-1} L_{jk} p_j$. It is a simple calculation that

$$\frac{\partial Q_{lk}}{\partial t_i} = [Q, L]_{lk}. \tag{3.8}$$

Comparing (3.8) with (1.4) we obtain $[Q, L] = [Q, Q^i_+]$. Since L and Q^i_+ are strictly upper matrices, this yields

$$L = Q^i_+. \tag{3.9}$$

We also need the equality

$$Q^i_{kl} h_k = Q^i_{lk} h_l. \tag{3.10}$$

Indeed

$$Q^i_{kl} h_k =< x^i \circ p_l, p_k >=< p_l, x^i p_k >=< x^p_l, p_k >=< p_l, x^i \circ p_k >= Q^i_{lk} h_l,$$

where $\circ$ stands for the action of the operator of multiplication by x^i in the basis $\{p_n\}_{n=0}^{\infty}$.

Now (3.5), (3.9) and (3.10) give

$$\frac{\partial h_k}{\partial t_i}\delta_{kl} = L_{lk} h_l + L_{kl} h_k - (Q^i)_{lk} h_l, \text{ or}$$

$$\begin{aligned} \frac{\partial}{\partial t_i} < p_k, p_l > &= < \sum_j L_{jk} p_j, p_l > + < \sum_j L_{jl} p_j, p_k > - < x^i p_k, p_l > \\ &= < \frac{\partial p_k}{\partial t_i}, p_l > + < p_k, \frac{\partial p_l}{\partial t_i} > - < x^i p_k, p_l > . \end{aligned}$$

From the fact that $\{p_n\}_{n=0}^{\infty}$ form a basis in $\mathbb{C}[x]$ we get

$$\frac{\partial}{\partial t_i} < P, Q >=< \frac{\partial P}{\partial t_i}, Q > + < P, \frac{\partial Q}{\partial t_i} > - < x^i P, Q >$$

for arbitrary polynomials P and Q. In particular,

$$\frac{\partial}{\partial t_i}\mu_n(t) = \frac{\partial}{\partial t_i} < x^n, 1 >= - < x^i x^n, 1 >= -\mu_{n+i}(t). \tag{3.11}$$

Before we state the main theorem we want to comment the structure of the solutions of (1.4). Suppose $A_n(0) \neq 0$ for every n. Then from (1.4) we get

$$\begin{aligned} \frac{\partial B_n}{\partial t_1} &= A_{n-1} - A_n, \\ \frac{\partial A_n}{\partial t_1} &= A_n(B_n - B_{n+1}). \end{aligned}$$

The last formulas show that $B_1(t)$ determines completely the solution $Q(t)$. In the case of $A_n = 0$ for some n the system (1.4) splits into two (or more) solutions to Toda lattice hierarchies. Then $B_1(t)$ determines the first solution.

On the other hand,

$$B_1(t) = -\frac{\partial}{\partial t_1} \log h_0(t) \tag{3.12}$$

and that is why $h_0(t)$ determines the solution.

Now we can state the main theorem of this section:

Theorem 3.1 *The only rational solutions to the Toda lattice hierarchy are given by*

$$\mu_0(t) = c\, S_n(-t_1 + \alpha_1, \ldots, -t_n + \alpha_n) \tag{3.13}$$

with some constants $c, \alpha_1, \ldots, \alpha_n$.

Proof. It is obvious that the solution given by μ_0 of the type (3.13) are rational.

Let $Q(t)$ be a rational solution to (1.4). Then $B_1(t)$ depends only on a finite number of $t_1, t_2, \ldots$. Let

$$B_1(t) = B_1(t_1, \ldots, t_N).$$

Then

$$Q(t) = Q(t_1, \ldots, t_N). \tag{3.14}$$

Put $f(t) = \mu_0(t)$. Then (3.11) gives

$$\frac{\partial^{|s|}}{\partial t^{(s)}} f(t) = \frac{\partial^{|\nu|}}{\partial t^{(\nu)}} f(t) \tag{3.15}$$

where $s = (s_1, \ldots, s_p)$ and $\nu = (\nu_1, \ldots, \nu_q)$ are multi-indeces with $\mid s \mid = s_1 + \ldots + s_p = \nu_1 + \ldots + \nu_q = \mid \nu \mid$ and $\frac{\partial}{\partial t^{(s)}} = \frac{\partial}{\partial t_{s_1}} \frac{\partial}{\partial t_{s_2}} \cdots \frac{\partial}{\partial t_{s_p}}$. Now equation (3.4) and (3.14) shows that $\log f(t)$ is a linear function of the variables $t_{N+1}, t_{N+2}, \ldots$, i.e.

$$\log f(t) = F(t_1, \ldots, t_N) + \sum_{k=N+1}^{\infty} c_k(t_1, \ldots, t_N) t_k. \tag{3.16}$$

Using (3.12) and the fact that B_1 does not depend on $t_{N+1}, t_{N+2}, \ldots$, etc. we conclude that c_k does not depend on t_1.

From (3.15) we get

$$\frac{\partial}{\partial t_{N+2}} f(t) = \frac{\partial^2}{\partial t_1 \partial t_{N+1}} f(t),$$

which means

$$c_{N+2} f(t) = c_{N+1} \frac{\partial}{\partial t_1} f(t)$$

or

$$c_{N+2} = c_{N+1} B_1(t).$$

Since c_{N+2} and c_{N+1} do not depend on t_1 we have either $c_{N+1} = c_{N+2} = 0$ and in a similar way $c_k = 0$ for $k > N$ or B_1 does not depend on t_1. The second asumption yields $A_1 = 0$ and that is why this case is not interesting. Thus we proved that $f(t)$ is a function of $t_1, t_2, \ldots, t_N$.

From

$$0 = \frac{\partial f}{\partial t_{N+i}} = \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_N} f \quad \text{for} \quad i = 1, 2, \ldots, N$$

we have

$$\frac{\partial}{\partial t_N} f = \text{const}.$$

Now it is a simple check to show that from (3.15) it follows that

$$\mu_0(t) = \sum_{k=0}^{N} c_k S_k(-t). \tag{3.17}$$

Without abuse of generosity we suppose $c_N \neq 0$. Then

$$\mu_0(t) = c \sum_{k=0}^{N} a_k S_k(-t) \text{ with } a_N = 1. \tag{3.18}$$

Consider

$$E = \exp((-t_1 + \alpha_1)x + (-t_2 + \alpha_2)x^2 + \ldots + (-t_N + \alpha_N)x^N - t_{N+1}x^{N+1} - \ldots).$$

we have

$$E = \sum_{k=0}^{\infty} S_k(-t_1 + \alpha_1, -t_2 + \alpha_2, \ldots, -t_N + \alpha_N, -t_{N+1}, \ldots)x^k. \tag{3.19}$$

On the other hand

$$\begin{aligned} E &= \exp(\alpha_1 x + \alpha_2 x^2 + \ldots, \alpha_N x^N) \exp(-V(x)) \\ &= \sum_{k=0}^{\infty} S_k(\alpha_1, \ldots, \alpha_N, 0, \ldots) \sum_{l=0}^{\infty} S_l(-t) x^{k+l}. \end{aligned} \tag{3.20}$$

Comparing the coefficients in front of x^N in (3.19) and (3.20) we get

$$S_N(-t_1 + \alpha_1, \ldots, -t_N + \alpha_N) = \sum_{s=0}^{N} S_{N-s}(\alpha_1, \ldots, \alpha_N) S_s(-t).$$

Equations $S_{N-s}(\alpha_1, \ldots, \alpha_N) = a_s,\ s = 0, \ldots, N$ determines uniquely $\alpha_1, \ldots, \alpha_N$ and hence there exists $\alpha_1, \ldots, \alpha_N; c$ such that

$$\mu_0(t) = c\, S_n(-t_1 + \alpha_1, \ldots, -t_n + \alpha_n)$$

which ends the proof of Theorem 3.1.

References

[1] Akhiezer N. I., The classical moment problem and some related questions in analysis, Hafner publishing company, New York, 1965.

[2] Bateman H., Erdelyi A., Higher transcedental functions, Mc Graw-Hill, New York, 1953.

[3] Dijkgraaf R., Verlinde E., Verlinde H., Loopequations and Virassoro constants in non-perturbative two-dimensional quantum gravity, Nucl. Phys. B 348 (1991) 435-456.

[4] Fukuma M.,Kawai H., Nagajama R., Continium Schwinger-Dyson equations and universal structures of two-dimensional quantum gravity, Int. J. Mod. Phys. A6 (1991) 1385-1406.

[5] Haine L., Horozov E., Toda orbits of Laguerre polynomials and representations of the Virasoro algebra, Bull. des Sci. Math. 1993. (to appear)

[6] Kac V. G.,Raina A.K., Bombay lectures on highest weight representation of infinite dimensional Lie algebras, Adv. serires in Math. Phys. vol.2, World Scientific, 1987.

[7] van Moerbeke P., Integrable foundations of string theory, Proc of the CIMPA Summer School (1991), to appear.