Partially and Fully Integrable Modules over Lie Superalgebras

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ABSTRACT. V. Kac and M. Wakimoto have observed in [**KW**] that for certain most natural affine Lie superalgebras $\mathfrak{g}$ like $\mathfrak{g} = sl\widehat{(m+n\varepsilon)}$ ($m \neq 0, 1$, $n \neq 0, 1$) or $\mathfrak{g} = osp\widehat{(m+n\varepsilon)}$ ($m \neq 1, 2$, $n \neq 1$), an irreducible highest weight $\mathfrak{g}$-module cannot be $\mathfrak{g}_0$-integrable unless it is the trivial module. Motivated by this fact, we introduce the notion of a partially integrable module for a large class of infinite-dimensional Lie superalgebras $\mathfrak{g}$. We also give a general definition of a highest weight module. We then prove an explicit criterion for partial integrability of irreducible highest weight $\mathfrak{g}$-modules. For the classical affine Lie superalgebras, which we consider in detail, this gives a stronger version of results of Kac-Wakimoto. However, our theorem can be applied to many other cases, in particular to $\mathfrak{g} = \mathbf{W}_{\mathrm{pol}}(m+n\varepsilon)$. We discuss this case briefly. Finally, for classical affine Lie superalgebras $\mathfrak{g}$ we announce a description of a class of irreducible $\mathfrak{g}$-modules which are $\mathfrak{g}_0$-integrable but are not highest weight modules. These are the loop modules introduced in the case of affine Lie algebras by V. Chari and A. Pressley.

Introduction

This paper originated in an attempt to understand the notion of integrability which V. Kac and M. Wakimoto, [**KW**], introduced for highest weight modules over classical affine Lie superalgebras $\mathfrak{g}$. The point is that those integrable $\mathfrak{g}$-modules are not integrable as $\mathfrak{g}_0$-modules [1)] . The following question arises: is it true that there is no reasonable class of $\mathfrak{g}$-modules which are integrable as $\mathfrak{g}_0$-modules? The answer is certainly *no* because for instance the loop modules (introduced by V. Chari and A. Pressley for affine Lie algebras, see [**CP**]) are also defined for $\mathfrak{g}$ and are $\mathfrak{g}_0$-integrable. However, loop modules are not highest weight modules. It turned out that in the case of the highest weight modules considered by Kac and Wakimoto we are actually dealing with the phenomenon of *partial integrability.* In this paper we give a definition of a partially integrable module for a general class of infinite-dimensional Lie superalgebras. This definition applies not only to highest weight modules, and moreover the notion of a highest weight module is not routine in our context. Therefore we also give a rather general definition of a highest weight module. Our main result is a criterion for partial integrability of irreducible highest weight modules. It generalizes a theorem proved earlier by V. Serganova and the second author, [**PS**], which characterizes finite-dimensional irreducible modules over a finite-dimensional Lie superalgebra.

In the second part of the paper we consider in detail various classical affine Lie superalgebras. The above theorem enables us to completely characterize certain

[1)] Throughout this paper the term "integrability" has the same meaning as in the book of V. Kac [**K2**]. The problem of whether the fully integrable or partially integrable modules we consider can be integrated to a group action could be addressed in a separate publication.

maximal families of line superalgebras (or "rank 1 subsuperalgebras") with respect to which a highest weight module can be integrable. We also discuss the case of the Lie superalgebra $\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon)$ of polynomial vector fields of m even and n odd indeterminates.

In the third part we introduce the category of bounded $\mathfrak{g}$-modules and announce an explicit criterion for integrability of irreducible modules in this category. As a corollary we obtain that all bounded $\mathfrak{g}_0$-integrable irreducible modules are loop modules. We intend to present these and other results in more detail in a forthcoming publication.

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Notation

The ground field is $\mathbb{C}$ and, if the opposite is not explicitly stated, all vector spaces are defined over $\mathbb{C}$ and are automatically assumed to be $\mathbb{Z}_2$-graded. All Lie superalgebras are also defined over $\mathbb{C}$. A right lower index $_0$ or $_1$ will always refer to $\mathbb{Z}_2$-grading. The upper index * denotes dual space and Π denotes the functor of parity change on vector spaces (Π acts also on the representations of any Lie superalgebra). The signs $\Subset$ or $\Supset$ denote semi-direct sum of Lie superalgebras. If a vector space $V = V_0 \oplus V_1$ has finite dimension, its dimension $\dim V$ is an element of the Clifford ring $\mathbb{Z}[\varepsilon]$, ε being an odd variable with $\varepsilon^2 = 1$. One has $\dim V = \dim V_0 + \dim(\Pi V_1)\cdot\varepsilon$.

1. Partially integrable highest weight modules: general theory

1.1. **A class of Lie superalgebras $\mathfrak{g}$.** Let $\mathfrak{g}$ be any complex Lie superalgebra, which has a nilpotent self-normalizing finite-dimensional Lie subsuperalgebra $\mathfrak{h}$ such that as $\mathfrak{h}$-module $\mathfrak{g}$ decomposes as

$$\mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Delta\subset\mathfrak{h}_0^*\setminus 0} \mathfrak{g}^{(\alpha)} \Big), \tag{1}$$

where for each α the linear space $\mathfrak{g}^{(\alpha)}$ is a direct sum of finite-dimensional $\mathfrak{h}$-modules whose composition factors over $\mathfrak{h}_0$ are $1-$ or $\varepsilon-$dimensional modules on which $\mathfrak{h}_0$ acts via $\alpha : \mathfrak{h}_0 \to \mathbb{C}$. The set Δ (i.e. the set of non-zero α's occuring in (1)) is by definition the set of *roots* of $\mathfrak{g}$. A root α is *even* iff $\mathfrak{g}_0^{(\alpha)} \neq 0$, and respectively *odd* iff $\mathfrak{g}_1^{(\alpha)} \neq 0$ (α may well be both even and odd). We denote the even roots by Δ_0 and the odd roots respectively by Δ_1. Clearly $\Delta = \Delta_0 \cup \Delta_1$. A 1-dimensional real subspace of $\mathfrak{h}_0^*$ will be called a *line of* $\mathfrak{g}$ if $l \cap \Delta \neq \emptyset$.

Any finite-dimensional Lie superalgebra (see [**PS**]), as well as any Kac-Moody Lie algebra, belongs to the above class. Our main examples (see 2.1) will be various affine Lie superalgebras. However, the polynomial versions of the infinite-dimensional Cartan series of Lie superalgebras, in particular $\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon) :=$

$\mathrm{der}(S^{\cdot}(\mathbb{C}^{m+n\varepsilon}))$ [2)], also belong to this class, see 2.2. In what follows we will assume the pair $\mathfrak{h} \subset \mathfrak{g}$ to be fixed and will call $\mathfrak{h}$ a *Cartan subsuperalgebra of* $\mathfrak{g}$.

$\mathfrak{C}$ will denote the category of $\mathfrak{g}$-modules V, such that as a $\mathfrak{h}$-module V decomposes as $\underset{\mu\in\mathfrak{h}_0^*}{\oplus} V^{(\mu)}$, each $V^{(\mu)}$ being a direct sum of finite-dimensional $\mathfrak{h}$-modules whose composition factors over $\mathfrak{h}_0$ are 1- or ε-dimensional $\mathfrak{h}$-modules on which $\mathfrak{h}_0$ acts via $\mu : \mathfrak{h}_0 \to \mathbb{C}$, and such that $\mu \neq 0$ at least for one μ. $V^{(\mu)}$ are the *generalized weight spaces* of V.

1.2. **Borel subsuperalgebras of $\mathfrak{g}$ and highest weight $\mathfrak{g}$-modules.** We start by defining triangular decompositions of Δ. Let F be a flag of maximal length which consists of real subspaces in $\mathfrak{h}_0^* : F = \{0 = F^0 \subset F^1 \subset F^2 \subset \cdots \subset F^{2n-1} \subset F^{2n} = \mathfrak{h}_0^*\}$, where $n = \dim_{\mathbb{C}} \mathfrak{h}_0$. Then using F we can split Δ into two non-intersecting sets as follows:

$$\begin{aligned} \Delta^+ &= ((F^{2n})^+ \cap \Delta) \sqcup ((F^{2n-1})^+ \cap \Delta) \sqcup \cdots \sqcup ((F^1)^+ \cap \Delta), \\ \Delta^- &= ((F^{2n})^- \cap \Delta) \sqcup ((F^{2n-1})^- \cap \Delta) \sqcup \cdots \sqcup ((F^1)^- \cap \Delta), \end{aligned} \tag{2}$$

where $(F^k)^+$ and $(F^k)^-$ denote respectively the two connected components of $F^k \backslash F^{k-1}$. The signs $+$ and $-$ are assigned to connected components arbitrarily, therefore F can determine in this way up to 2^{2n} different decompositions. Each decomposition (2) is by definition a *triangular decomposition of* Δ. Given a triangular decomposition (2), by reversing the signs we obtain the *opposite triangular decomposition*.

Clearly, a triangular decomposition does not necessarily determine the flag F, i.e. different flags can define the same triangular decomposition. Moreover, sometimes the decomposition depends actually only on a subflag of F. For example, if $\dim \mathfrak{g} < \infty$, any triangular decomposition is actually determined by a real regular subspace P of $\mathfrak{h}_0^*$ (regular means that $P \cap \Delta = \emptyset$) of dimension $2n-1$. In other words, the reader will check straightforwardly that in this case for any triangular decomposition there exists a regular $2n-1$-dimensional real subspace P so that this given decomposition is determined by an arbitrary flag F of the form $\{0 \subset F^1 \subset F^2 \subset \cdots \subset F^{2n-1} = P \subset F^{2n} = \mathfrak{h}_0^*\}$. For any triangular decomposition ($\mathfrak{g}$ being possibly infinite-dimensional) the length of a shortest flag by which it is actually determined, is a combinatorial invariant of that decomposition.

A Lie subsuperalgebra $\mathfrak{b} = \mathfrak{h} \ni\!\!\!\!+ (\underset{\alpha\in\Delta^+}{\oplus} \mathfrak{g}^{(\alpha)})$, corresponding to some triangular decomposition $\Delta = \Delta^+ \sqcup \Delta^-$, is by definition a *Borel subsuperalgebra of* $\mathfrak{g}$. We set $\mathfrak{n}^\pm = \underset{\alpha\in\Delta^\pm}{\oplus} \mathfrak{g}^{(\alpha)}$. Then $\mathfrak{b} = \mathfrak{h} \ni\!\!\!\!+ \mathfrak{n}^+$. When we need to refer to $\mathfrak{n}^\pm$ for a given $\mathfrak{b}$ we will write $\mathfrak{n}^\pm(\mathfrak{b})$. The Borel subsuperalgebra $\mathfrak{b}^- = \mathfrak{h} \ni\!\!\!\!+ \mathfrak{n}^-$ (defined by the opposite triangular decomposition) is the Borel subsuperalgebra *opposite* to $\mathfrak{b}$. Any Borel subsuperalgebra defines a partial order on $\mathfrak{h}_0^*$:

$$\mu \leq_{\mathfrak{b}} \eta \quad \Leftrightarrow \quad \eta = \mu + \sum_i \alpha_i, \;\; \alpha_i \in \Delta^+, \;\; \text{or} \;\; \mu = \eta.$$

[2)] In [**PS**] and [**P**] only finite-dimensional Lie superalgebras are considered and $\mathrm{der} S^{\cdot}(\mathbb{C}^{n\varepsilon})$ is denoted simply by $\mathbf{W}(n)$. In the notation of the present paper $\mathrm{der}\, S^{\cdot}(\mathbb{C}^{n\varepsilon}) = \mathbf{W}(n\varepsilon) = \mathbf{W}_{\mathrm{pol}}(n\varepsilon)$.

Two Borel subsuperalgebras $\mathfrak{b}^1$ and $\mathfrak{b}^2$ are *adjacent* if they correspond respectively to triangular decompositions $(\Delta^+)^1 \sqcup (\Delta^-)^1$ and $(\Delta^+)^2 \sqcup (\Delta^-)^2$ such that

$$(\Delta^+)^1 \backslash ((\Delta^+)^1 \cap l) = (\Delta^+)^2 \backslash ((\Delta^+)^2 \cap l) \tag{3}$$

for some line l of $\mathfrak{g}$. A line l of $\mathfrak{g}$ is *simple* for a Borel subsuperalgebra $\mathfrak{b}$ if one can find a Borel subsuperalgebra $\mathfrak{b}'$ so that $\mathfrak{b}$ and $\mathfrak{b}'$ are adjacent and (3) holds. A sequence of Borel subsuperalgebras $\dots, \mathfrak{b}^{i_1}, \dots, \mathfrak{b}^{i_k}, \dots$ is a *chain* if $\mathfrak{b}^{i_n}$ and $\mathfrak{b}^{i_{n+1}}$ are adjacent for every $n \in \mathbb{Z}$. Given a chain $\dots, \mathfrak{b}^{i_1}, \dots, \mathfrak{b}^{i_k}, \dots$, it determines a sequence of lines $\dots, l^{i_1}, \dots, l^{i_{k-1}}, \dots$, where l^{i_j} is the unique simple line for both $\mathfrak{b}^{i_j}$ and $\mathfrak{b}^{i_{j+1}}$. If $\dim \mathfrak{g} < \infty$, any pair of Borel subsuperalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ (which contains the fixed Cartan subsuperalgebra $\mathfrak{h}$) can be connected by a finite chain $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'$. This of course is not true when $\dim \mathfrak{g} = \infty$.

If $\mathfrak{b}$ is a Borel subsuperalgebra, its finite-dimensional irreducible modules are in bijective correspondence with the finite-dimensional irreducible $\mathfrak{h}$-modules. Indeed, if ν is a finite-dimensional irreducible $\mathfrak{b}$-module, one shows exactly as in Proposition 2 of [**PS**] that there is a non-zero $\mathfrak{h}$-submodule $\tilde{\nu}$ of ν on which $\mathfrak{n}^+$ acts trivially. Since ν is irreducible and $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}$, $\nu = \tilde{\nu}$. Conversely, if ν is an irreducible $\mathfrak{h}$-module, one can always endow it with a $\mathfrak{b}$-action by letting $\mathfrak{n}^+$ act trivially on ν.

Irreducible finite-dimensional modules over any nilpotent Lie superalgebra have been described by V. Kac in [**K1**]. Kac's result, applied to $\mathfrak{h}$, states that any linear function $\lambda \in \mathfrak{h}_0^*$, such that $\lambda|_{[\mathfrak{h}_0,\mathfrak{h}_0]} = 0$, determines a finite-dimensional irreducible $\mathfrak{h}$-module ν'_λ. Moreover, ν'_λ is the induced module $U(\mathfrak{h}) \otimes_{U(\mathfrak{p})} \check{\lambda}$, where $\mathfrak{p}$ is a Lie subsuperalgebra of $\mathfrak{h}$ with $\mathfrak{p} \supset \mathfrak{h}_\lambda := \mathfrak{h}_0 \oplus (\mathfrak{h}_\lambda)_1$, for $(\mathfrak{h}_\lambda)_1 := \{h_1 \in \mathfrak{h}_1 | \lambda([h_1, h'_1]) = 0 \ \forall h_1 \in \mathfrak{h}_1\}$, and such that $\mathfrak{p}$ is a maximal subsuperalgebra for which λ (considered as a 1-dimensional $\mathfrak{h}_0$-module) extends to a 1-dimensional $\mathfrak{p}$-module $\check{\lambda}$. Furthermore, obviously $\nu''_\lambda = \Pi\nu'_\lambda$ is also a well-defined finite-dimensional irreducible $\mathfrak{g}$-module (which may or may not be isomorphic to ν'_λ), and the second part of Kac's result states that any finite-dimensional irreducible $\mathfrak{h}$-module is isomorphic to ν'_λ or ν''_λ for some $\lambda \in \mathfrak{h}_0^*$, $\lambda|_{[\mathfrak{h}_0,\mathfrak{h}_0]} = 0$. In what follows ν_λ will denote a finite-dimensional irreducible $\mathfrak{b}$-module which as $\mathfrak{h}$-module is isomorphic either to ν'_λ or to ν''_λ. An element $\lambda \in \mathfrak{h}_0^*$ is by definition a *weight of* $\mathfrak{g}$ iff ν_λ is well-defined, i.e. iff $\lambda|_{[\mathfrak{h}_0,\mathfrak{h}_0]} = 0$.

If ν_λ is an irreducible finite-dimensional $\mathfrak{b}$-module, then the induced $\mathfrak{g}$-module

$$\tilde{V}_{\mathfrak{b}}(\nu_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \nu_\lambda$$

is by definition the *Verma module with* $\mathfrak{b}$*-highest weight space* ν_λ. It is crucial that $\tilde{V}_{\mathfrak{b}}(\nu_\lambda)$ is an object of the category $\mathfrak{C}$ (for $\lambda \neq 0$ in the special case when $\mathfrak{b} = \mathfrak{g}$) and that $\tilde{V}_{\mathfrak{b}}(\nu_\lambda)$ has a unique maximal proper $\mathfrak{g}$-submodule. The first statement is obvious and the second statement is a straightforward corollary of the definition of a Borel subsuperalgebra. This implies that $\tilde{V}_{\mathfrak{b}}(\nu_\lambda)$ has a unique irreducible $\mathfrak{g}$-factormodule which we shall denote by $V_{\mathfrak{b}}(\nu_\lambda)$. $V_{\mathfrak{b}}(\nu_\lambda)$ is by definition the *irreducible* $\mathfrak{b}$*-highest weight* $\mathfrak{g}$*-module with highest weight space* ν_λ. More generally, any factormodule of $\tilde{V}_{\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}$*-highest weight module with highest weight space* ν_λ. In what follows $\mathcal{O}_{\mathfrak{b}}$ will denote the category of $\mathfrak{g}$-modules which admit a finite filtration whose factors are $\mathfrak{b}$-highest weight modules, and $\mathcal{O}_{\mathfrak{b}}^{\mathfrak{C}}$ will denote the subcategory of $\mathcal{O}_{\mathfrak{b}}$ whose objects are also objects of $\mathfrak{C}$.

1.3. **Line subsuperalgebras and integrability of $\mathfrak{g}$-modules.** To every line l of $\mathfrak{g}$ we can assign a *line subsuperalgebra* $\mathfrak{g}^l$, which is by definition the Lie subsuperalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}^{(\alpha)}$ for $\alpha \in l \cap \Delta$. We will call a line l *finite* (respectively *infinite*) iff $\dim \mathfrak{g}^l < \infty$ (respectively $\dim \mathfrak{g}^l = \infty$). The results of [**PS**] enable us to classify all line superalgebras $\mathfrak{g}^l$ for finite l. Indeed, the following Proposition is an immediate corollary of Proposition 3 in [**PS**].

PROPOSITION 1. *If* $\dim \mathfrak{g}^l < \infty$*, then there are the following alternatives:*

(i) $\mathfrak{g}^l$ *is nilpotent;*
(ii) $\mathfrak{g}^l \simeq \mathfrak{r} \Subset sl(2)$*;*
(iii) $\mathfrak{g}^l \simeq \mathfrak{r} \Subset osp(1+2\varepsilon)$*,*

where in (ii) *and* (iii) $\mathfrak{r}$ *is the radical of* $\mathfrak{g}^l$ *and this radical is nilpotent.* □

We will say that a finite line l is of *type* (i), (ii), or (iii), respectively if (i), (ii), or (iii) holds.

If $\mathfrak{b}$ is a Borel subsuperalgebra of $\mathfrak{g}$, then for any l we set $\mathfrak{b}^l = \mathfrak{h}^l \Subset (\mathfrak{n}^+)^l$, where $\mathfrak{h}^l = \mathfrak{h} \cap \mathfrak{g}^l$ and $(\mathfrak{n}^+)^l = \mathfrak{n}^+ \cap \mathfrak{g}^l$. However $\mathfrak{b}^l$ is not necessarily a Borel subsuperalgebra of $\mathfrak{g}^l$. For instance, if l is finite, $\mathfrak{b}^l$ is a Borel subsuperalgebra of $\mathfrak{g}^l$ iff l is of types (ii) or (iii). For l of type (i), $\mathfrak{g}^l$ coincides with its own Cartan subsuperalgebra. Nevertheless, for any finite l, we can define a $\mathfrak{g}^l$-module $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$, λ^l being the restriction of a weight $\lambda \in \mathfrak{h}_0^*$ to $\mathfrak{h}_0^l$, where $\mathfrak{h}^l = \mathfrak{g}^l \cap \mathfrak{h}$. Indeed, if l is of type (ii) or (iii), then $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ is the irreducible $\mathfrak{b}^l$-highest weight $\mathfrak{g}^l$-module with highest weight space ν_{λ^l}, ν_{λ^l} being an irreducible $\mathfrak{h}^l$-module corresponding to λ^l. If l is of type (i), $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ is the unique (up to isomorphism) finite-dimensional irreducible $\mathfrak{g}^l$-module which contains ν_{λ^l} as $\mathfrak{h}^l$-submodule and (when considered as $\mathfrak{g}_0^l$-module) consists of a single generalized weight space of weight λ', λ' being the extension of λ^l by zero on $(\mathfrak{n}^{\pm})_0^l$ (where $(\mathfrak{n}^{\pm})^l = \mathfrak{n}^{\pm} \cap \mathfrak{g}^l$). For l of type (ii) or (iii), $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ exists for any λ. But in order $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ to be finite-dimensional, λ^l must satisfy an additional condition (for instance, when $\mathfrak{g}^l \simeq sl(2)$ λ^l must be integral and dominant). For l of type (i), $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ (being defined as a finite-dimensional module) exists iff $\lambda|_{[\mathfrak{g}_0^l, \mathfrak{g}_0^l]} = 0$. Therefore, for any finite l the *finite-dimensionality condition on* λ^l *with respect to* $\mathfrak{b}^l$ is a condition on λ^l which ensures that $\dim V_{\mathfrak{b}^l}(\nu_{\lambda^l}) < \infty$ (or that $V_{\mathfrak{b}^l}(\nu_{\lambda^l})$ exists for l of type (i)).

If V is an object of $\mathfrak{C}$ and l is a line of $\mathfrak{g}$, we will call V $l-$*integrable* iff every $g^\alpha \in \mathfrak{g}^{(\alpha)}$ for $\alpha \in l \cap \Delta$ acts locally nilpotently on V. If L is an arbitrary subset of the set of lines of $\mathfrak{g}$, then V is $L-$*integrable* whenever it is l-integrable for all $l \in L$. We will call V is *partially integrable* iff it is L-integrable for some proper subset L of the set of all lines of $\mathfrak{g}$.

If dim $\mathfrak{g} < \infty$, and V is an irreducible $\mathfrak{g}$-module, then V is l-integrable for all lines l of $\mathfrak{g}$ iff dim $V < \infty$. If $\mathfrak{g}$ is a Kac-Moody Lie algebra, the accepted notion of integrability, see for instance [**K2**], is in our terminology L_f-integrability, L_f being the set of all finite lines of $\mathfrak{g}$. One of the starting points of the present paper was the observation of Kac-Wakimoto, see [**KW**], that for the affine Lie superalgebra $sl(\widehat{m+n\varepsilon})$, $m, n \geq 2$, see 2.1.1 below, a highest weight module with non-zero highest weight can not be L_f-integrable and is therefore at most partially integrable.

Let us observe also that any subcategory $\mathfrak{C}'$ of $\mathfrak{C}$ defines a partial order on the set of subsets of lines of $\mathfrak{g}$. Indeed, if L_1 and L_2 are two sets of lines, then we put

$$L_1 \leq^{\mathfrak{C}'} L_2$$

iff each L_2-integrable module V' of $\mathfrak{C}'$ is necessarily L_1-integrable. A straightforward checking confirms that $\leq^{\mathfrak{C}'}$ is a well-defined partial order. For a fixed $\mathfrak{C}'$ the maximal elements with respect to the order $\leq^{\mathfrak{C}'}$ can be of considerable interest and in Theorem 2 in 2.1.3 we will describe explicitly some $\leq^{\mathcal{O}_{\mathfrak{b}}^{\mathfrak{C}}}$-maximal sets.[3)]

1.4. **Partial integrability of highest weight modules.** Our main result in the first part of the paper is a general criterion for partial integrability of highest weight $\mathfrak{g}$-modules. It applies to certain sets of finite lines L which we call connected. By definition, a set of lines of $\mathfrak{g}$ L is *connected* if there exists a set of Borel subsuperalgebras $B(L)$ with the properties:

- any two Borel subsuperalgebras $\mathfrak{b}', \mathfrak{b}'' \in B(L)$ can be connected by a finite chain $\mathfrak{b}' = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}''$, such that the corresponding sequence of lines $l^1, l^2, \dots, l^{k-1}$ belongs to L,

- for any l in L there exists $\mathfrak{b}(l) \in B(L)$, such that l is a simple line for $\mathfrak{b}(l)$.

If L is connected, $B(L)$ is not necessarily unique. This becomes clear when considering examples. If the contrary is not explicitly stated, when writing $B(L)$ below we will mean an arbitrary but fixed choice of $B(L)$. If $\mathfrak{g}$ is finite-dimensional, the set of all lines of $\mathfrak{g}$ is connected. If $\mathfrak{g}$ is any of the affine Lie superalgebras defined in 2.1.1, the set L_f of all finite lines of $\mathfrak{g}$ is connected.

If now $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'$ is a chain of Borel subsuperalgebras from $B(L)$, so that the corresponding sequence of lines $l^1, \dots, l^{k-1}$ belongs to L, and $\lambda \in \mathfrak{h}_0^*$ is a weight of $\mathfrak{g}$, there is a natural way to define a weight λ^i for each $i = 1, \dots, k$ so that $\lambda^1 = \lambda$. The definition is inductive. Assume that $\lambda^1 = \lambda$ and that λ^i is defined up to $i = j$. The line l^j is either of type (i), or respectively of types (ii) and (iii). Consider first the case when l^j is of type (i). If λ^j satisfies the condition $\lambda^j|_{[\mathfrak{g}_0^{l^j}, \mathfrak{g}_0^{l^j}]} = 0$, then λ^{j+1} is by definition the $\mathfrak{b}^{j+1} \cap (\mathfrak{h} + \mathfrak{g}^{l^j})$-highest weight of an irreducible $\mathfrak{h} + \mathfrak{g}^{l^j}$-module with $\mathfrak{b}^j \cap (\mathfrak{h} + \mathfrak{g}^{l^j})$-highest weight λ^j. According to Proposition 6 in [**PS**], in this case

$$\lambda^{j+1} = \lambda^j + \mathrm{ch} S^{\cdot}((\mathfrak{n}_1^-)^{l^j} / f^-_{\lambda^{l^j}}), \tag{4}$$

$S^{\cdot}$ denoting supersymmetric algebra (here Grassmann algebra since $(\mathfrak{n}_1^-)^{l^j}$ is a

3) It may be worth to note that although a $\mathfrak{g}$-module V from the category $\mathcal{O}_{\mathfrak{b}}$ need not be l-integrable for any line l of $\mathfrak{g}$, V is automatically $\mathfrak{b}$-integrable, i.e. V is $l^{\mathfrak{b}}$-integrable for all lines $l^{\mathfrak{b}}$ of the Lie superalgebra $\mathfrak{b}$.

purely odd space) and $f^-_{\lambda^{l^j}}$, being the left kernel of the pairing

$$(\mathfrak{n}_1^-)^{l^j} \times (\mathfrak{n}_1^+)^{l^j} \xrightarrow{[\cdot,\cdot]} \mathfrak{g}_0^{l^j} \to \mathfrak{h}_0 \cap \mathfrak{g}_0^{l^j} \xrightarrow{\lambda^{l^j}} \mathbb{C}\ .$$

If $\lambda^j|_{[\mathfrak{g}_0^{l^j},\mathfrak{g}_0^{l^j}]} \neq 0$, formula (4) still makes sense and therefore, when l^{j+1} is of type (i), one can simply define λ^{j+1} by formula (4). If now l^j is of type (ii) or (iii), we define $\lambda_{\mathfrak{b}}^{\mathfrak{b}^{j+1}}$ to be the weight of $\mathfrak{g}$ obtained by applying to $\lambda_{\mathfrak{b}}^{\mathfrak{b}^j}$ the only non-trivial Weyl group reflection of the Lie algebra $(\mathfrak{h} + \mathfrak{g}^{l^j})_0$. (For any finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}'$, the Weyl group is the subgroup in $\mathrm{Aut}_{\mathbb{C}}\mathfrak{h}'$ generated by reflections along each of the roots of the semi-simple part of $\mathfrak{g}$, by definition each such reflection being identical on the intersection of $\mathfrak{h}'$ with the radical of the Lie algebra. The semi-simple part of $(\mathfrak{h} + \mathfrak{g}^{l^j})_0$ is $sl(2)$ and the Weyl group is $\mathbb{Z}_2$). In this way λ^i is defined for any i and in particular for $i = k$. In some cases it is clear that λ^k depends only on λ, on $\mathfrak{b}$, and on $\mathfrak{b}'$, but not on the chain which connects $\mathfrak{b}$ and $\mathfrak{b}'$. If the latter is true, we set $\lambda^k = \lambda_{\mathfrak{b}}^{\mathfrak{b}'}$ and say that $\lambda_{\mathfrak{b}}^{\mathfrak{b}'}$ is well-defined.

We are now able to formulate our main result about integrability for highest weight modules. Informally speaking, it states that, if the adjoint representation of $\mathfrak{g}$ is L-integrable and L is a connected set of finite lines, the condition of L-integrability of $V_{\mathfrak{b}}(\nu_\lambda)$ for any $\mathfrak{b} \in B(L)$ (and any ν_λ) can be localized to each individual line l in L, after passing to a suitable Borel subsuperalgebra $\mathfrak{b}(l)$ for which l is simple. Here is the precise statement.

THEOREM 1. *Let $L \subset L_f$ be connected set of lines of $\mathfrak{g}$, such that $\mathfrak{g}$ is L-integrable as $\mathfrak{g}$-module, and let $V_{\mathfrak{b}}(\nu_\lambda)$ be an irreducible $\mathfrak{b}$-highest weight module for $\mathfrak{b} \in B(L)$. Then $V_{\mathfrak{b}}(\nu_\lambda)$ is L-integrable, iff, for every $\mathfrak{b}' \in B(L)$ and for any simple line $l' \in L$ of $\mathfrak{b}'$, $\lambda_{\mathfrak{b}}^{\mathfrak{b}'}$ is well-defined and the weight $(\lambda_{\mathfrak{b}}^{\mathfrak{b}'})^{l'}$ of $\mathfrak{g}^{l'}$ satisfies the finite-dimensionality condition with respect to $(\mathfrak{b}')^{l'}$.*

PROOF. It is similar to the proof of Theorem 1 in [**PS**], the same idea being applied here in a more general situation. Assume first that $V_{\mathfrak{b}}(\nu_\lambda)$ is L-integrable. We claim that then $V_{\mathfrak{b}}(\nu_\lambda) \simeq V_{\mathfrak{b}'}(\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}'}})$ for any $\mathfrak{b}' \in B(L)$, i.e. that $V_{\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}'$-highest weight module with highest weight space $\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}'}}$. Indeed, let $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'$ be a chain connecting $\mathfrak{b}$ and $\mathfrak{b}'$, and let $l^1, \dots, l^{k-1}$ be the corresponding sequence of lines. $V_{\mathfrak{b}}(\nu_\lambda)$ is l^1-integrable (since $l^1 \in L$) and thus the $U(\mathfrak{g}^{l^1})$-submodule $U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda$ of $V_{\mathfrak{b}}(\nu_\lambda)$ is of finite dimension. Therefore if $\lambda(l^1)$ is a weight, such that $V_{\mathfrak{b}}(\nu_\lambda)^{(\lambda(l^1))} \cap (U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda) \neq \emptyset$ and $\lambda(l^1)$ is maximal with respect to the partial order $\leq_{(\mathfrak{b}^-)^{l^1}}$ on $\mathfrak{h}_0^*$ induced by $(\mathfrak{b}^-)^{l^1}$ [4)], then $\mathfrak{n}^+(\mathfrak{b}^2)$ annihilates $V_{\mathfrak{b}}(\nu_\lambda)^{(\lambda(l^1))} \cap (U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda)$. Indeed, $\mathfrak{n}^-(\mathfrak{b}) \cap \mathfrak{g}^{l^1}$ annihilates $V_{\mathfrak{b}}(\nu_\lambda)^{(\lambda(l^1))} \cap (U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda)$ because of the maximality of $\lambda(l^1)$, while $\mathfrak{g}^{(\alpha)}$ for $\alpha \in \Delta^+ \backslash (\Delta^+ \cap l^1)$ annihilate $V_{\mathfrak{b}}(\nu_\lambda)^{(\lambda(l^1))} \cap (U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda)$ because of the simplicity of the line l^1 for $\mathfrak{b}$, i.e. because of the fact that $[g_\alpha^+, g_{l^1}^-] \in \bigoplus_{\beta \in \Delta^+ \backslash (\Delta^+ \cap l^1)} \mathfrak{g}^{(\beta)}$ for any $\alpha \in \Delta^+ \backslash (\Delta^+ \cap l^1)$,

---

4) by definition $\mu \leq_{(\mathfrak{b}^-)^{l^1}} \eta \ \Leftrightarrow\ \eta = \mu + \sum_i \alpha_i,\ \ \alpha_i \in \Delta^- \cap l^1$, or $\mu = \eta$.

$g_\alpha^+ \in \mathfrak{g}^{(\alpha)}$, and any $\mathfrak{g}_{l^1}^- \in \mathfrak{n}^-(\mathfrak{b}) \cap \mathfrak{g}^{l^1}$. This means that an irreducible $\mathfrak{h}$-submodule of $V_\mathfrak{b}(\nu_\lambda)^{(\lambda(l^1))} \cap (U((\mathfrak{n}^-)^{l^1}) \cdot \nu_\lambda)$ is a $\mathfrak{b}^2$-submodule of $V_\mathfrak{b}(\nu_\lambda)$, i.e. that $V_\mathfrak{b}(\nu_\lambda)$ is a $\mathfrak{b}^2$-highest weight $\mathfrak{g}$-module, or equivalently that $V_\mathfrak{b}(\nu_\lambda) \simeq V_{\mathfrak{b}^2}(\nu_{\lambda(l^1)})$ for some irreducible $\mathfrak{b}^2$-module $\nu_{\lambda(l^1)}$. But therefore $\lambda(l^1)$ equals λ^2 and the latter equals $\lambda_\mathfrak{b}^{\mathfrak{b}^2}$, i.e. in particular $\lambda_\mathfrak{b}^{\mathfrak{b}^2}$ is well-defined. Thus we have simply

$$V_\mathfrak{b}(\nu_\lambda) \simeq V_{\mathfrak{b}^2}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^2}})$$

for a certain $\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^2}}$. Continuing the process now, i.e. applying the same argument to $V_{\mathfrak{b}^2}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^2}})$ and to the line l^2, etc. (since $V_{\mathfrak{b}^2}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^2}})$ is l^2-integrable, etc.), we obtain a chain of $\mathfrak{g}$-isomorphisms

$$V_\mathfrak{b}(\nu_\lambda) \simeq V_{\mathfrak{b}^2}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^2}}) \simeq V_{\mathfrak{b}^3}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}^3}}) \simeq \cdots \simeq V_{\mathfrak{b}'}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}'}}).$$

This implies in particular that $\lambda_\mathfrak{b}^{\mathfrak{b}'}$ is well-defined and that $V_{\mathfrak{b}'}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}'}})$ is L-integrable. If now $l' \in L$ is a simple line of $\mathfrak{b}'$, the $\mathfrak{g}^{l'}$-module $U((\mathfrak{n}^-)^{l'}) \cdot \nu_{\lambda_\mathfrak{b}^{\mathfrak{b}'}}$ is finite-dimensional because of the l'-integrability of $V_{\mathfrak{b}'}(\nu_{\lambda_\mathfrak{b}^{\mathfrak{b}'}})$ and thus $\lambda_\mathfrak{b}^{\mathfrak{b}'}$ necessarily satisfies the finite-dimensionality condition with respect to $(\mathfrak{b}')^{l'}$. In this way we have proved that the L-integrability of $V_\mathfrak{b}(\nu_\lambda)$ implies that, for each $\mathfrak{b}' \in B(L)$ and for any of its simple lines $l' \in L$, $\lambda_\mathfrak{b}^{\mathfrak{b}'}$ is well-defined and satisfies the finite-dimensionality condition with respect to $(\mathfrak{b}')^{l'}$.

It remains to establish the opposite, i.e. that if $\lambda_\mathfrak{b}^{\mathfrak{b}'}$ is well-defined for all $\mathfrak{b}' \in B(L)$ and if all respective conditions on finite-dimensionality for the simple lines of all $\mathfrak{b}'$ are satisfied, then $V_\mathfrak{b}(\nu_\lambda)$ is L-integrable. Let us show first that, for a simple line l of $\mathfrak{b}$, $V_\mathfrak{b}(\nu_\lambda)$ is l-integrable whenever λ^l satisfies the finite-dimensionality condition with respect to $\mathfrak{b}^l$. We start with the observation that, if this latter condition is satisfied, one has $U((\mathfrak{n}^-)^l) \cdot \nu_\lambda < \infty$. (Indeed, otherwise the maximal proper submodule of $U((\mathfrak{n}^-)^l) \cdot \nu_\lambda$ with zero intersection with ν_λ would generate a proper $\mathfrak{g}$-submodule of $V_\mathfrak{b}(\nu_\lambda)$ and this would contradict to the irreducibility of $V_\mathfrak{b}(\nu_\lambda)$). Moreover, there is a natural surjection of $U(\mathfrak{g}^l)$-modules

$$\mu : U((\bigoplus_{\substack{\alpha\in\Delta^- \\ \alpha\notin l^1}} \mathfrak{g}^{(\alpha)}) \Subset \mathfrak{h}) \otimes_{U(\mathfrak{g}^l)} U((\mathfrak{n}^-)^l) \cdot \nu_\lambda \to V_\mathfrak{b}(\nu_\lambda) \tag{5}$$

such that

$$\mu(g' \otimes g'' \cdot v) = g' \cdot g'' \cdot v$$

for $g' \in U((\bigoplus_{\substack{\alpha\in\Delta^- \\ \alpha\notin l^1}} \mathfrak{g}^{(\alpha)}) \Subset \mathfrak{h})$, $g'' = U(\mathfrak{g}^l)$. Therefore in order to establish the l-integrability of $V_\mathfrak{b}(\nu_\lambda)$, we need to establish the l-integrability of the tensor product in (5). But, being finite-dimensional, $U((\mathfrak{n}^-)^l) \cdot \nu_\lambda$ is obviously l-integrable, and, as the reader will verify, the l-integrability of $U((\bigoplus_{\substack{\alpha\in\Delta^- \\ \alpha\notin l^1}} \mathfrak{g}^{(\alpha)}) \Subset \mathfrak{h})$ follows from the l-integrability of $\mathfrak{g}$. Since l-integrability is preserved by taking tensor product, we obtain that $V_\mathfrak{b}(\nu_\lambda)$ is l-integrable, being a $\mathfrak{g}^l$-factormodule of a l-integrable $\mathfrak{g}^l$-module.

Let now l' be any line in L. Let $\mathfrak{b}'(l')$ be a Borel subsuperalgebra for which l' is simple. Connect $\mathfrak{b}$ and $\mathfrak{b}'(l')$ by a chain $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'(l')$ such that the corresponding sequence of lines $l^1, l^2, \dots, l^{k-1}$ belongs to L. The crucial point is to note that, since $(\lambda_{\mathfrak{b}}^{\mathfrak{b}^i})^{l^i}$ satisfies the finite-dimensionality condition with respect to $\mathfrak{b}^i$ for each $i = 1, \dots, k-1$, we obtain by applying consecutively the same arguments as in the first part of the proof that $V_{\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}^{i+1}$-highest weight $\mathfrak{g}$-module for $i = 1, \dots, k-1$, i.e. in particular that $V_{\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}'$-highest weight $\mathfrak{g}$-module. (Indeed, at the i-th step we conclude that $U((\mathfrak{n}^-)^{l^i}) \cdot \nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}^i}} < \infty$ because $(\lambda_{\mathfrak{b}}^{\mathfrak{b}^i})^{l^i}$ satisfies the finite-dimensionality condition with respect to $\mathfrak{b}^i$, and then we show as in the beginning of the proof that this gives $V_{\mathfrak{b}^i}(\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}^i}}) \simeq V_{\mathfrak{b}^{i+1}}(\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}^{i+1}}})$.) But then $V_{\mathfrak{b}}(\nu_\lambda) \simeq V_{\mathfrak{b}'}(\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}'}})$, and the above argument ensures the l'-integrability of $V_{\mathfrak{b}'}(\nu_{\lambda_{\mathfrak{b}}^{\mathfrak{b}'}})$. The proof of Theorem 1 is complete. $\square$

The above arguments imply also

COROLLARY 1. *Let L be a connected set of finite lines of $\mathfrak{g}$, such that $\mathfrak{g}$ is a L-integrable module. Fix $\mathfrak{b} \in B(L)$. Then for a $\mathfrak{b}$-highest weight irreducible $\mathfrak{g}$-module V the following two conditions are equivalent:*

- *V is L-integrable;*
- *V is a $\mathfrak{b}'$-highest weight $\mathfrak{g}$-module for any $\mathfrak{b}' \in B(L)$.* $\square$

2. Applications: the cases of classical affine Lie superalgebras and of $\mathbf{W}_{\mathbf{pol}}(m+n\varepsilon)$

2.1. **Affine Lie superalgebras.**

2.1.1. **Definitions.** In what follows $\mathfrak{k}$ denotes a finite-dimensional Lie superalgebra and $\Sigma = \Sigma_0 \cup \Sigma_1$ is the set of roots of $\mathfrak{k}$. If t is an even formal variable and $D = t\frac{d}{dt}$, we define the Lie superalgebras $\mathfrak{k}^{loop}$ by setting

$$\mathfrak{k}^{loop} = \mathfrak{k} \otimes \mathbb{C}[t, t^{-1}] \in \mathbb{C}D.$$

The bracket in $\mathfrak{k}^{loop}$ is determined by the relation

$$[x \otimes t^m + rD, y \otimes t^n + sD] = [x, y] \otimes t^{n+m} + nry \otimes t^n - msx \otimes t^m \tag{6}$$

for $x, y \in \mathfrak{k}$, $r, s \in \mathbb{C}$. Let $\mathfrak{h}_{\mathfrak{k}}$ be a (fixed) Cartan subsuperalgebra of $\mathfrak{k}$. Then $\mathfrak{h}^{loop} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathbb{C}D$ is a Cartan superalgebra of $\mathfrak{k}^{loop}$.

Suppose that $(\cdot|\cdot)$ is an even invariant bilinear form on $\mathfrak{k}$. Set $\hat{\mathfrak{k}} = \mathfrak{k}^{loop} \ni \mathbb{C}K$, where K is an even formal variable. $\hat{\mathfrak{k}}$ is a central extension of $\mathfrak{k}^{loop}$ and its bracket is determined by the relation

$$\begin{aligned}[x \otimes t^m + pK + rD,\ y \otimes t^n + uK + sD] = [x, y] \otimes t^{n+m} + nry \otimes t^n - \\ - msx \otimes t^m + m\delta_{m,-n}(x|y)K.\end{aligned} \tag{7}$$

$\hat{\mathfrak{h}} = \mathfrak{h}^{loop} \oplus \mathbb{C}K$ is a Cartan subsuperalgebra of $\hat{\mathfrak{k}}_0$ and $(\cdot|\cdot)$ extends to an even invariant form $\widehat{(\cdot|\cdot)}$ on $\hat{\mathfrak{k}}$, where

$$(x \otimes \widehat{t^n | y} \otimes t^m) = \delta_{n,-m}(x|y), \quad (x \widehat{\otimes t^n} | K) = (x \widehat{\otimes t^m} | D) = 0,$$

$$\widehat{(K|K)} = \widehat{(D|D)} = 0, \ \ \widehat{(K|D)} = 1.$$

$\widehat{(\cdot|\cdot)}$ is non-degenerate whenever $(\cdot|\cdot)$ is non-degenerate. An odd invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{k}$ also defines a central extension of $\mathfrak{k}^{loop}$. The relation which determines its bracket is the same as (7), but in this case K is an odd formal variable. The resulting algebra will be denoted by $\hat{\mathfrak{k}}$ as well.

We will consider also the following twisted affine Lie superalgebras:

$$\begin{aligned} \mathfrak{k}^{loop}_{tw} &= (\mathfrak{k}_0 \otimes \mathbb{C}[t^2, t^{-2}] \oplus \mathfrak{k}_1 \otimes t\mathbb{C}[t^2, t^{-2}]) \in\!\!\!\!\in \mathbb{C}D \subset \mathfrak{k}^{loop}, \\ \hat{\mathfrak{k}}_{tw} &= ((\mathfrak{k}_0 \otimes \mathbb{C}[t^2, t^{-2}] \oplus \mathfrak{k}_1 \otimes t\mathbb{C}[t^2, t^{-2}]) \in\!\!\!\!\in \mathbb{C}D) \ni\!\!\!\!\ni \mathbb{C}K \subset \hat{\mathfrak{k}}. \end{aligned}$$

Below we shall assume that $\mathfrak{k}$ is a simple classical Lie superalgebra and that $(\cdot|\cdot)$ is non-degenerate. This means that $(\cdot|\cdot)$ is even when $\mathfrak{g} \simeq sl(m + n\varepsilon)$ $(m \neq n)$, $psl(n+n\varepsilon)$ $(n > 1)$, $osp(m+2n\varepsilon)$, $\mathbf{G}(3)$, $\mathbf{F}(4)$ and that $(\cdot|\cdot)$ is odd when $\mathfrak{g} \simeq psq(n)$ $(n > 2)$. (In all other cases $\mathfrak{k}$ admits no non-degenerate invariant form). $\mathfrak{g}$ will be one of the Lie superalgebras defined above and $\mathfrak{h}$ will be its respective Cartan subsuperalgebra.

Define $c \in (\mathfrak{h}^{loop})^*$ by setting $c|_{\mathfrak{h}_\mathfrak{k}} = 0$, $c(D) = 1$. Putting $c(K) = 0$, we extend c to a linear function on $\hat{\mathfrak{h}}$ when $\hat{\mathfrak{h}}$ is defined. We define also $d \in \hat{\mathfrak{h}}^*$ by setting $d|_{\mathfrak{h}^{loop}} = 0$, $d(K) = 1$. Γ will denote the real vector space spanned by the roots of $\mathfrak{k}$ (i.e. $\Gamma = \mathbb{R}\Sigma$), and $\hat{\Gamma} := \Gamma \oplus \mathbb{R}c$.

Here are the root systems of all $\mathfrak{g}$ considered. If $\mathfrak{g} = \mathfrak{k}^{loop}$ or $\mathfrak{g} = \hat{\mathfrak{k}}$, then

$$\begin{aligned} \Delta \ &= \{\alpha + nc \mid \alpha \in \Sigma, \ n \in \mathbb{Z}\} \cup \{nc \mid n \in \mathbb{Z}\backslash\{0\}\}, \\ \Delta_0 &= \{\alpha + nc \mid \alpha \in \Sigma_0, \ n \in \mathbb{Z}\} \cup \{nc \mid n \in \mathbb{Z}\backslash\{0\}\}, \\ \Delta_1 &= \begin{cases} \{\alpha + nc \mid \alpha \in \Sigma_1, \ n \in \mathbb{Z}\} \ \ \text{when} \ \ (\mathfrak{h}_\mathfrak{k})_1 = 0 \\ \{\alpha + nc \mid \alpha \in \Sigma_1, \ n \in \mathbb{Z}\} \cup \{nc \mid n \in \mathbb{Z}\backslash\{0\}\} \ \ \text{when} \ \ (\mathfrak{h}_\mathfrak{k})_1 \neq 0. \end{cases} \end{aligned}$$

If $\mathfrak{g} = \mathfrak{k}^{loop}_{tw}$ or $\hat{\mathfrak{k}}_{tw}$, then

$$\begin{aligned} \Delta \ &= \{\alpha{+}2nc \mid \alpha{\in}\Sigma_0, \ n{\in}\mathbb{Z}\} \cup \{\alpha{+}(2n+1)c \mid \alpha{\in}\Sigma_1, \ n{\in}\mathbb{Z}\} \cup \{nc \mid n{\in}\mathbb{Z}\backslash\{0\}\}, \\ \Delta_0 &= \{\alpha{+}2nc \mid \alpha{\in}\Sigma_0, \ n{\in}\mathbb{Z}\} \cup \{2nc \mid n \in \mathbb{Z}\backslash\{0\}\}, \\ \Delta_1 &= \begin{cases} \{\alpha + (2n+1)c \mid \alpha \in \Sigma_1, \ n{\in}\mathbb{Z}\} \ \text{when} \ (\mathfrak{h}_\mathfrak{k})_1 = 0 \\ \{\alpha + (2n+1)c \mid \alpha \in \Sigma_1, \ n{\in}\mathbb{Z}\} \cup \{(2n+1)c \mid n{\in}\mathbb{Z}\backslash\{0\}\} \ \text{when} \ (\mathfrak{h}_\mathfrak{k})_1 \neq 0. \end{cases} \end{aligned}$$

In all cases we define the *imaginary line* of $\mathfrak{g}$ l^{im} as $l^{im} := \mathbb{R}c$.

2.1.2. **Standard Borel subsuperalgebras.** For all $\mathfrak{g}$ introduced in 2.1.1, the length of the shortest flag by which a given triangular decomposition is determined is less or equal to 2. Indeed, it is a straightforward observation that any triangular decomposition is either determined by a regular real hyperplane in $\mathfrak{h}_0^*$ (i.e. there exists a hyperplane P so that the given decomposition is determined by any flag $0 \subset F^1 \subset \cdots \subset F^{2n-1} = P \subset F^{2n} = \ hg_0^*$) or by a flag $l^{im} \subset P$ where $(P\backslash(P \cap l^{im})) \cap \Delta = \emptyset$ (i.e. there exists a hyperplane P with $P \supset l^{im}$, $(P\backslash(P \cap l^{im})) \cap \Delta = \emptyset$, so that the given decomposition is determined by any flag $0 \subset F^1 = l^{im} \subset \cdots \subset F^{2n-1} = P \subset F^{2n} = \mathfrak{h}_0^*$). A Borel subsuperalgebra of $\mathfrak{g}$ is *standard* if it can be determined by a regular hyperplane. A Borel subsuperalgebra that cannot be determined by a regular hyperplane is *imaginary*.

If $\mathfrak{b}$ is standard, there always exits a real linear functional φ on $\widehat{\Gamma}$ with $\Delta^{\pm} = \{\alpha \in \Delta \mid \varphi(\alpha) \gtrless 0\}$ and $\varphi(c) \neq 0$. Indeed, for a standard $\mathfrak{b}$, $\widehat{\Gamma} \cap P$ is a hyperplane in $\widehat{\Gamma}$ and φ is determined uniquely by the requirements $\varphi(\Delta^+) > 0$, $\varphi(\widehat{\Gamma} \cap P) = 0$. Conversely, if $\varphi \in (\widehat{\Gamma})^*$ and $\varphi(\alpha) \neq 0 \, \forall \alpha \in \Delta$, then $\Delta^+_\varphi \sqcup \Delta^-_\varphi$ is a standard triangular decomposition, where $\Delta^{\pm}_\varphi = \{\alpha \in \Delta \mid \varphi(\alpha) \gtrless 0\}$. After multiplying φ by $\frac{1}{|\varphi(c)|}$ we can assume that $\varphi(c) = \pm 1$. In what follows $\varphi_{\mathfrak{b}}$ will denote such a normalized functional corresponding to a standard Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$.

We call a subset $\{\beta_1, \dots, \beta_s\}$ of Δ *a basis of* Δ if every $\alpha \in \Delta$ can be represented in a unique way as $\alpha = \sum_{i=1}^s c_i \beta_i$ for $c_i \in \mathbb{Z}$, and all c_i are either non-negative or non-positive. Every basis of Δ defines a Borel subsuperalgebra: its roots are all $\alpha = \sum_{i=1}^s c_i \beta_i$ with non-negative c_i's. The opposite is not true since the imaginary Borel subsuperalgebras of $\mathfrak{g}$ do not admit a basis. However any standard Borel subsuperalgebra is determined by a basis. More precisely, we have

PROPOSITION 2. *Every standard Borel subsuperalgebra of $\mathfrak{g}$ is determined by a unique basis of Δ. Any basis of Δ is of the form*

$$\alpha_1 + n_1 c, \alpha_2 + n_2 c, \dots, \alpha_s + n_s c, -\alpha + n_{s+1} c, \tag{8}$$

where $\alpha_1, \dots, \alpha_s$ is a basis of Σ, $\alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_s\alpha_s$ is the longest root of Σ with respect to this basis, and

- $b_1 n_1 + b_2 n_2 + \dots + b_s n_s + n_{s+1} = \begin{cases} 1 & \text{if } c \in \Delta^+ \\ -1 & \text{if } c \in \Delta^- \end{cases}$

  for $\mathfrak{g} = \hat{\mathfrak{k}},\ \mathfrak{g} = \mathfrak{k}^{loop},\ \mathfrak{g} = \widehat{psq(r)}_{tw},\ \mathfrak{g} = psq(r)^{loop}_{tw}$,

- $b_1 n_1 + b_2 n_2 + \dots + b_s n_s + n_{s+1} = \begin{cases} 2 & \text{if } c \in \Delta^+ \\ -2 & \text{if } c \in \Delta^- \end{cases}$

  for $\mathfrak{g} = \hat{\mathfrak{k}}_{tw},\ \mathfrak{g} = \mathfrak{k}^{loop}_{tw},\ \mathfrak{k} \neq psq(n)$.

SKETCH OF THE PROOF. A straightforward way to establish the Proposition is to consider each case for $\mathfrak{g}$ separately. Here we will present the proof for $\mathfrak{g} = \hat{\mathfrak{k}}$, where $\mathfrak{k} = osp((2r+1)+2n\varepsilon)$, $r \geq 2$. One has $\Sigma = \{\pm\varepsilon_i, \pm\delta_p, \pm 2\delta_p, \pm\varepsilon_i \pm \varepsilon_j, \pm\delta_p \pm \delta_q, \pm\varepsilon_i \pm \delta_p \mid 1 \leq i \neq j \leq r,\ 1 \leq p \neq q \leq n\}$, see [**K1**]. It is clear that every set of the form (8) is a basis of Δ and that it determines a standard Borel subsuperalgebra of $\mathfrak{g}$. We need to prove the converse. Let $\mathfrak{b}$ be a standard Borel subsuperalgebra of $\mathfrak{g}$. Consider the numbers $\{\varphi_{\mathfrak{b}}(\pm\varepsilon_i)\} := \varphi_{\mathfrak{b}}(\pm\varepsilon_i) - [\varphi_{\mathfrak{b}}(\pm\varepsilon_i)]$ and $\{\varphi_{\mathfrak{b}}(\pm\delta_p)\} := \varphi_{\mathfrak{b}}(\pm\delta_p) - [\varphi_{\mathfrak{b}}(\pm\delta_p)]$, where $1 \leq i \leq r$, $1 \leq p \leq n$, and $[x]$ denotes the integer part of x. One checks, using the fact that $\varphi_{\mathfrak{b}}(\varepsilon_i)$ and $\varphi_{\mathfrak{b}}(\delta_p)$ are never integers, that $\{\varphi_{\mathfrak{b}}(\varepsilon_i)\} + \{\varphi_{\mathfrak{b}}(-\varepsilon_i)\} = \{\varphi_{\mathfrak{b}}(\delta_p)\} + \{\varphi_{\mathfrak{b}}(-\delta_p)\} = 1$. Therefore setting

$$\begin{aligned}
a_1 &= \begin{cases} \varepsilon_1 & \text{if} \quad \{\varphi_{\mathfrak{b}}(\varepsilon_1)\} \leq \frac{1}{2} \\ -\varepsilon_1 & \text{if} \quad \{\varphi_{\mathfrak{b}}(-\varepsilon_1)\} < \frac{1}{2} \end{cases}, \\
&\vdots \\
a_n &= \begin{cases} \varepsilon_n & \text{if} \quad \{\varphi_{\mathfrak{b}}(\varepsilon_n)\} \leq \frac{1}{2} \\ -\varepsilon_n & \text{if} \quad \{\varphi_{\mathfrak{b}}(-\varepsilon_n)\} < \frac{1}{2} \end{cases}, \\
a_{n+1} &= \begin{cases} \delta_1 & \text{if} \quad \{\varphi_{\mathfrak{b}}(\delta_1)\} < \frac{1}{2} \\ -\delta_1 & \text{if} \quad \{\varphi_{\mathfrak{b}}(-\delta_1)\} < \frac{1}{2} \end{cases}, \\
&\vdots \\
a_{n+r} &= \begin{cases} \delta_r & \text{if} \quad \{\varphi_{\mathfrak{b}}(\delta_r)\} < \frac{1}{2} \\ -\delta_r & \text{if} \quad \{\varphi_{\mathfrak{b}}(-\delta_r)\} < \frac{1}{2} \end{cases},
\end{aligned}$$

we can assume, after suitably reordering the sequence $a_1, \ldots, a_{n+r}$, that

$$\{\varphi_{\mathfrak{b}}(a_1)\} < \{\varphi_{\mathfrak{b}}(a_2)\} < \cdots < \{\varphi_{\mathfrak{b}}(a_{r+n})\} \leq \frac{1}{2}.$$

The reader will verify then immediately that

$$a_{r+n} - a_{r+n-1}, a_{r+n-1} - a_{r+n-2}, \ldots, a_2 - a_1, a_1$$

is a basis of Σ. Denoting this basis by $\alpha_1, \ldots, \alpha_{r+n}$, and setting $n_i = [\varphi_{\mathfrak{b}}(\alpha_i)]$, $i = 1, \ldots, r+n$, we obtain the desired basis of $\mathfrak{b}$ by formula (8). $\square$

All Borel subsuperalgebras considered in 2.1.3 and 2.1.4 will be assumed standard. The unique basis of Δ which determines $\mathfrak{b}$ will be referred to as the basis of $\mathfrak{b}$ and its elements are by definition the simple roots of $\mathfrak{b}$.

2.1.3. **Partial integrability and maximal families of lines.** Note that in all cases considered $\mathfrak{g}_0$ has certain distinguished Lie subalgebras. Let the semisimple part of $\mathfrak{k}_0$ be isomorphic to $\overset{i_{\mathfrak{k}}}{\underset{i=1}{\oplus}} \mathfrak{k}^i$, where $1 \leq i_{\mathfrak{k}} \leq 3$. (For instance $i_{\mathfrak{k}} = 2$ when $\mathfrak{k} = sl(m+n\varepsilon)$, $m > 1$ $n > 1$, and in this case $\mathfrak{k}^1 \simeq sl(m)$, $\mathfrak{k}^2 \simeq sl(n)$; $i_{\mathfrak{k}} = 3$ only for $\mathbf{D}(2,1;\alpha)$ and then $\mathfrak{k}^1 \simeq \mathfrak{k}^2 \simeq \mathfrak{k}^3 \simeq sl(2)$). We will consider $\mathfrak{k}^i$ as Lie subalgebra of $\mathfrak{k}_0$. Then for $\mathfrak{g} = \mathfrak{k}^{loop}$ we set $\mathfrak{g}_0^i = (\mathfrak{k}^i)^{loop}$, for $\mathfrak{g} = \hat{\mathfrak{k}}$ we set $\mathfrak{g}_0^i = \hat{\mathfrak{k}^i}$ (where $\hat{\mathfrak{k}^i}$ is defined via the restriction of $(\cdot|\cdot)$ to $\mathfrak{k}^i$), and for $\mathfrak{g} = \mathfrak{k}_{tw}^{loop}$ or $\mathfrak{g} = \hat{\mathfrak{k}}_{tw}$ we set respectively $\mathfrak{g}_0^i = (\mathfrak{k}^i)^{loop} \cap \mathfrak{g}$ and $\mathfrak{g}_0^i = \hat{\mathfrak{k}^i} \cap \mathfrak{g}$. By Δ^i we denote the roots of $\mathfrak{g}_0^i$ considered as a subset of Δ. Furthermore $L_f^i = \{l \in L_f \mid l \cap \Delta^i \neq \emptyset\}$, $L^{odd} := \{l \in L_f \mid \mathfrak{g}^l \simeq sl(1+\varepsilon)\}$. For every standard Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$, $\mathfrak{b} \cap \mathfrak{g}_0^i$ is a Borel subalgebra of $\mathfrak{g}_0^i$, where $1 \leq i \leq i_{\mathfrak{k}}$. Finally, let Γ^i be the real vector space spanned by Δ^i.

The following Theorem gives an explicit description of all $\leq^{\mathcal{O}_{\mathfrak{b}}^{\mathfrak{C}}}$-maximal sets of lines and of corresponding sets $B(L_m)$. In this way, given $\mathfrak{b}$ we know explicitly its maximal set L_m, and conversely, given L_m we know all Borel subsuperalgebras $\mathfrak{b}'$ to which it corresponds (see Corollary 1 in 1.4).

THEOREM 2. *Let $\mathfrak{b}$ be a fixed Borel subsuperalgebra of $\mathfrak{g}$ and $\mathfrak{C}' = \mathcal{O}_{\mathfrak{b}}^{\mathfrak{C}}$.*

a) *Let $\mathfrak{g} = \widehat{sl(m+\varepsilon)}$, $\widehat{sl(1+n\varepsilon)}$, $\widehat{sl(m+\varepsilon)}_{tw}$, $\widehat{sl(1+n\varepsilon)}_{tw}$, or $\mathfrak{g} = \widehat{osp(m+2n\varepsilon)}$, $\widehat{osp(m+2n\varepsilon)}_{tw}$ for $m = 1, 2$. Then $L_m = L_f$ is the only set of lines which* *is maximal with respect to* $\leq^{\mathfrak{C}'}$*, and* $B_m(L_m) = \{\mathfrak{b}' \,|\, \varphi_{\mathfrak{b}'}(c) = \varphi_{\mathfrak{b}}(c)\}$ *is the largest among all sets* $B(L_m)$ *such that* $\mathfrak{b} \in B(L_m)$.

b) *Let* $\mathfrak{g} = \widehat{sl(m+n\varepsilon)}$, $\widehat{sl(m+n\varepsilon)}_{tw}$ *for* $m,n \geq 2$, *or* $\mathfrak{g} = \widehat{osp(m+2n\varepsilon)}$, $\widehat{osp(m+2n\varepsilon)}_{tw}$ *for* $m > 2$, *or* $\mathfrak{g} = \widehat{\mathbf{F}(4)}$, $\widehat{\mathbf{F}(4)}_{tw}$, *or* $\mathfrak{g} = \widehat{\mathbf{G}(3)}$, $\widehat{\mathbf{G}(3)}_{tw}$. *Then a family of lines* L_m *is maximal with respect to* $\leq^{\mathfrak{C}'}$ *iff it is of the form* $L' = L^1 \cup L^2 \cup L^{odd}$*, where either* $L^1 = L^1_f$ *and* $L^2 = \Theta^2 \cap L^2_f$*, or* $L^2 = L^2_f$ *and* $L^1 = \Theta^1 \cap L^1_f$*, and* Θ^i [5] *is a real hyperplane in* Γ^i *which admits a basis of elements from* Δ^i*. If* $B_m(L_m) = \{\mathfrak{b}' \mid \varphi_{\mathfrak{b}'}(c) = \varphi_{\mathfrak{b}}(c)$ *and* Θ^2 *(resp.* Θ^1*) contains* $\dim \Theta^2$ *(resp.* $\dim \Theta^1$*) simple roots of* $\mathfrak{b}' \cap \mathfrak{g}^2_0$ *(resp.* $\mathfrak{b}' \cap \mathfrak{g}^1_0)\}$ *for* $L^1 = L^1_f$ *(resp.* $L^2 = L^2_f$*), then* $B_m(L_m)$ *is the largest among all sets* $B(L_m)$ *such that* $\mathfrak{b} \in B(L_m)$.

c) *Let* $\mathfrak{g} = sl(m+n\varepsilon)^{loop}$, $sl(m+n\varepsilon)^{loop}_{tw}$, $osp(m+2n\varepsilon)^{loop}$, $osp(m+2n\varepsilon)^{loop}_{tw}$, $\mathbf{D}(2,1;\alpha)^{loop}$, $\mathbf{D}(2,1;\alpha)^{loop}_{tw}$, $\mathbf{F}(4)^{loop}$, $\mathbf{F}(4)^{loop}_{tw}$, $\mathbf{G}(3)^{loop}$, $\mathbf{G}(3)^{loop}_{tw}$, $\widehat{psq(n)}$, $\widehat{psq(n)}_{tw}$, $psq(n)^{loop}$, $psq(n)^{loop}_{tw}$. *Then a family of lines* L_m *is maximal with respect to* $\leq^{\mathfrak{C}'}$ *iff it is of the form* $L'' = (\cup_{i=1}^{i_{\mathfrak{k}}} L^i) \cup L^{odd}$*, where* $L^i = \Theta^i \cap L^i_f$*,* Θ^i *being a real hyperplane in* Γ^i *which admits a basis of elements from* Δ^i*. If* $B_m(L_m) = \{\mathfrak{b}' \mid \varphi_{\mathfrak{b}'}(c) = \varphi_{\mathfrak{b}}(c)$ *and* Θ^i *contains* $\dim \Theta^i$ *simple roots of* $\mathfrak{b}' \cap \mathfrak{g}^i_0$ *for* $1 \leq i \leq i_{\mathfrak{k}}$ *and there exists a chain* $\mathfrak{b} = \mathfrak{b}^1, \cdots, \mathfrak{b}^k = \mathfrak{b}'$ *with corresponding lines* $l^1, \ldots, l^{k-1}$ *in* $L_m\}$*, then* $B_m(L_m)$ *is the largest among all sets* $B(L_m)$ *such that* $\mathfrak{b} \in B(L_m)$.

Sketch of Proof. The proof is based on

Proposition 3. *Let* $\mathfrak{l}$ *be a Kac-Moody Lie algebra whose Dynkin diagram is of finite or affine type (see* [**K2**]*, Chap. 4),* $\mathfrak{b}_{\mathfrak{l}}$ *be a standard Borel subalgebra in* $\mathfrak{l}$*, and* λ *be a weight of* $\mathfrak{l}$*. If* $L^{\lambda}_{\mathfrak{b}_{\mathfrak{l}}}$ *is the set of all finite lines of* $\mathfrak{l}$ *for which* $V_{\mathfrak{b}_{\mathfrak{l}}}(\nu_\lambda)$ *is* $L^{\lambda}_{\mathfrak{b}_{\mathfrak{l}}}$*-integrable, then there exists a vector subspace* $\Xi^{\lambda}_{\mathfrak{b}_{\mathfrak{l}}}$ *in the real vector space spanned by the roots of* $\mathfrak{l}$ *so that* $\ell \in L^{\lambda}_{\mathfrak{b}_{\mathfrak{l}}}$ *iff* $\ell \subset \Xi^{\lambda}_{\mathfrak{b}_{\mathfrak{l}}}$.

We will present the proof of this Proposition in a forthcoming article. In the present paper we restrict ourselves to deducing the statement of Theorem 2 from Proposition 3.

a) First of all, for any $\mathfrak{g}$ considered in the Theorem (i.e. for $\mathfrak{g}$ as in a), b), or c)), if $V_{\mathfrak{b}}(\nu_\lambda)$ is l^{im}-integrable, then necessarily $\lambda = 0$, i.e. $V_{\mathfrak{b}}(\nu_\lambda)$ is not in $\mathfrak{C}$. Therefore $L_m \subset L_f$. In order to prove that $L_m = L_f$ it suffices to prove that in this case there exists at least one weight $\lambda^0 \neq 0$ for which $V_{\mathfrak{b}}(\nu_{\lambda^0})$ is L_f-integrable. In 2.1.4 we will write down explicitly the necessary and sufficient condition for the L_f-integrability of $V_{\mathfrak{b}}(\nu_\lambda)$ for any λ, which will give $L_m = L_f$ immediately. The statement about $B_m(L_m)$ in a) is obvious.

b) Assuming that $V_{\mathfrak{b}}(\nu_{\lambda^0})$ is L_f-integrable for some λ^0, we obtain that the irreducible $\mathfrak{g}^i_0$-module with $\mathfrak{b} \cap \mathfrak{g}^i_0$-highest weight $\lambda^0|_{\mathfrak{h} \cap \mathfrak{g}^i_0}$ is L^i_f-integrable for $i = 1, 2$. But this implies $\lambda^0(K) \geq 0$ and $\lambda^0(K) \leq 0$, i.e. $\lambda^0(K) = 0$, which gives $\lambda^0 = 0$, (see [**K2**], Chap. 10). Therefore $L_m \neq L_f$. If we assume that $V_{\mathfrak{b}}(\nu_{\lambda^0})$ is L-integrable

[5] In the cases considered $i_{\mathfrak{k}} = 2$ and thus $i = 1, 2$.

for some non-empty $L \subset L_f$, Proposition 3 implies that L is contained in one of the sets $L^1 \cup L^2 \cup L^{odd}$.

Let now $L' = L^1 \cup L^2 \cup L^{odd}$ be a fixed set of finite lines as in the statement of the Theorem. We need to prove that there exists $\lambda^0 \neq 0$ for which $V_{\mathfrak{b}}(\nu_{\lambda^0})$ is L'-integrable. For each $\mathfrak{g}$ considered such a λ^0 can be written down explicitly. Note that L' is connected (this is a straightforward checking), so we can apply Theorem 1 to L'. Let's show how to find λ^0 for $\mathfrak{g} = sl(\widehat{m+n\varepsilon})$, $n > 1$.

Fix L' as $L^1 \cup L^2 \cup L^{odd}$ where $L^1 = L^1_f$. It is straightforward to verify that there exists a Borel subsuperalgebra $\mathfrak{b}'$ of $\mathfrak{g} = sl(\widehat{m+n\varepsilon})$ with the following properties:

- there is a chain $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'$ such that, if $\ell^1, \dots, \ell^{k-1}$ is the corresponding set of lines, then $\mathfrak{g}^{\ell^i} \simeq sl(1+\varepsilon)$ for $i = 1, \dots, k-1$,
- the simple roots of $\mathfrak{b}'$ are of the form $\varepsilon_{i_1} - \varepsilon_{i_2} + s_1 c, \dots, \varepsilon_{i_m} - \delta_{j_1} + s_m c$, $\delta_{j_1} - \delta_{j_2} + t_1 c, \dots, \delta_{j_n} - \varepsilon_{i_1} + t_n c$,[6] where $(i_1, \dots, i_m)$ is a permutation of $(1, \dots, m)$ and $(j_1, \dots, j_n)$ is a permutation of $(1, \dots, n)$, $s_1 + \dots + s_m + t_1 + \dots + t_n = 1$, and $\delta_{j_p} - \delta_{j_{p+1}} + t_p c \in \Theta^2$ for $p = 1, \dots, n-1$.

The first property implies that $V_{\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}'$-highest weight module for any λ (since $V_{\mathfrak{b}}(\nu_\lambda)$ is automatically ℓ-integrable for any ℓ with $\mathfrak{g}^\ell \simeq sl(1+\varepsilon)$), and therefore it is enough to find a non-zero weight $(\lambda^0)'$ for which $V_{\mathfrak{b}'}(\nu_{(\lambda^0)'})$ is L_m-integrable. If λ' is such that

$$(\lambda', \varepsilon_i - \delta_j) + r\lambda'(K) \notin \mathbb{Z} \text{ for every } 1 \leq i \leq m,\ 1 \leq j \leq n,\ r \in \mathbb{Z}, \tag{9}$$

one verifies using Theorem 1 that the L_m-integrability of $V_{\mathfrak{b}'}(\nu_{\lambda'})$ is equivalent to the following conditions on λ':

$$\begin{array}{ll} (\lambda',\ \varepsilon_{i_p} - \varepsilon_{i_{p+1}} + s_p c) \in \mathbb{Z}_+, & p = 1, \dots, m-1; \\ (\lambda',\ \delta_{j_q} - \delta_{j_{q+1}} + t_q c) \in \mathbb{Z}_-, & q = 1, \dots, n-1; \\ (\lambda'',\ \varepsilon_{i_m} - \varepsilon_{i_1} + (s_m + t_1 + \cdot + t_n)c) \in \mathbb{Z}_+, & \text{where} \\ \quad \lambda'' = \lambda' - (\varepsilon_{i_m} - \delta_{j_1} + s_m c) - (\varepsilon_{i_m} - \delta_{j_2} + (s_m + t_1)c) - \cdots & \\ \quad \cdots - (\varepsilon_{i_m} - \delta_{j_n} + (s_m + t_1 + \dots + t_{n-1})c). & \end{array} \tag{10}$$

It is an elementary computation to check that (9) and (10) have a non-zero solution $(\lambda^0)'$.

The statement about $B_m(L_m)$ follows from the observation that $B_m(L_m)$ is the largest among all sets of Borel subsuperalgebras which contain $\mathfrak{b}$ and such that if $\mathfrak{b}' \in B_m(L_m)$ and $\mathfrak{b}' = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}''$ is a chain with corresponding lines $\ell^1, \dots, \ell^{k-1}$, where $\ell^i \in L_m$ for $i = 1, \dots, k-1$, then $\mathfrak{b}'' \in B_m(L_m)$. This proves b).

c) In this case one proceeds exactly as in b), see also 2.1.4. □

2.1.4. **A finite algorithm which determines whether $V_{\mathfrak{b}}(\nu_\lambda)$ is L_m-integrable.** If one applies directly Theorem 1 to $V_{\mathfrak{b}}(\nu_\lambda)$ one is faced with checking the finite-dimensionality conditions for infinitely many line subsuperalgebras. However, for the Lie superalgebras considered in Theorem 2 it is not hard to write down

[6] $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n$ denotes here the dual basis of a standard basis in the Cartan subalgebra of $gl(m+n\varepsilon)$; an explicit form of Σ is $\{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_l,\ \varepsilon_i - \delta_k \mid 1 \leq i \neq j \leq m,\ 1 \leq k \neq l \leq n\}$.

a finite algorithm which checks the L_m-integrability of $V_{\mathfrak{b}}(\nu_\lambda)$. We leave it to the reader to formally prove this algorithm in each case by using Theorem 1.

a) Assume that Theorem 2 a) holds.

- Let $\mathfrak{g} = \widehat{sl(m+\varepsilon)}$, $\widehat{sl(1+n\varepsilon)}$, $\widehat{sl(m+\varepsilon)}_{tw}$, $\widehat{sl(1+n\varepsilon)}_{tw}$, $\widehat{osp(2+2n\varepsilon)}$, $\widehat{osp(2+2n\varepsilon)}_{tw}$. If $\alpha_1, \dots, \alpha_N, \alpha_{N+1}$ are the simple roots of $\mathfrak{b}$ (where $N = m$ for $\mathfrak{g} = \widehat{sl(m+\varepsilon)}$, $\widehat{sl(m+\varepsilon)}_{tw}$, and $N = n$ for $\mathfrak{g} = \widehat{sl(1+n\varepsilon)}$, $\widehat{sl(1+n\varepsilon)}_{tw}$, $\widehat{osp(2+2n\varepsilon)}$, $\widehat{osp(2+2n\varepsilon)}_{tw}$), without loss of generality we can assume that $\mathfrak{g}^{\mathbb{R}\alpha_N} \simeq \mathfrak{g}^{\mathbb{R}\alpha_{N+1}} \simeq sl(1+\varepsilon)$. Then $V_{\mathfrak{b}}(\nu_\lambda)$ is integrable iff

$$2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}_+ \text{ for } 1 \le i \le N-1$$

and

$$\begin{cases} 2(\lambda, \alpha_N + \alpha_{N+1})/(\alpha_N + \alpha_{N+1}, \alpha_N + \alpha_{N+1}) \in \mathbb{Z}_+ & \text{when } (\lambda, \alpha_N) = 0 \\ 2(\lambda - \alpha_N, \alpha_N + \alpha_{N+1})/(\alpha_N + \alpha_{N+1}, \alpha_N + \alpha_{N+1}) \in \mathbb{Z}_+ & \text{when } (\lambda, \alpha_N) \ne 0. \end{cases}$$

- Let $\mathfrak{g} = \widehat{osp(1+2n\varepsilon)}$, $\mathfrak{g} = \widehat{osp(1+2n\varepsilon)}_{tw}$. Since in this case $\mathfrak{g}^\ell \simeq sl(2)$ or $\mathfrak{g}^\ell \simeq osp(1+2\varepsilon)$ for any $\ell \in L_f$, $V_{\mathfrak{b}}(\nu_\lambda)$ is L_f-integrable iff

$$2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_+ \text{ for every even root } \alpha \text{ of } \mathfrak{b}\,.$$

b) Assume that Theorem 2 b) holds and that $L^1 = L^1_f$, $L^2 = \Theta^2 \cap L^2_f$. There exists a chain $\mathfrak{b} = \mathfrak{b}^1, \dots, \mathfrak{b}^k = \mathfrak{b}'$ with $\mathfrak{g}^{\ell^i} \simeq sl(1+\varepsilon)$ for $i = 1, \dots, k-1$ ($\ell^1, \dots, \ell^{k-1}$ being the corresponding sequence of lines), and such that the basis of $\mathfrak{b}'$ contains all simple roots of $\mathfrak{b}' \cap \mathfrak{g}^2_0$ which are in Θ^2 and all simple roots of $\mathfrak{b}' \cap \mathfrak{g}^1_0$ except one or two.

- Suppose that there is a unique simple root α of $\mathfrak{b}' \cap \mathfrak{g}'_0$ which does not belong to the basis of $\mathfrak{b}'$. Then there exists a chain $\mathfrak{b}' = \mathfrak{b}^1, \dots, \mathfrak{b}^p = \mathfrak{b}''$ so that α is a simple root of $\mathfrak{b}''$ and $\mathfrak{g}^{\ell^i} \simeq sl(1+\varepsilon)$ for $i = 1, \dots, p-1$, $\ell'^1, \dots, \ell'^{p-1}$ being the corresponding sequence of lines. $V_{\mathfrak{b}}(\nu_\lambda)$ is L_m-integrable iff

$$\begin{aligned} &2(\lambda^{\mathfrak{b}'}_{\mathfrak{b}}, \gamma)/(\gamma, \gamma) \in \mathbb{Z}_+ \quad \text{for every even root } \gamma \text{ of } \mathfrak{b}', \\ &2(\lambda^{\mathfrak{b}''}_{\mathfrak{b}}, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_+; \end{aligned}$$

- Suppose that there are two simple roots α and β of $\mathfrak{b}' \cap \mathfrak{g}^1_0$ which do not belong to the basis of $\mathfrak{b}'$. Let $\mathfrak{b}''$ and $\mathfrak{b}'''$ be Borel subsuperalgebras of $\mathfrak{g}$ such that there exist two chains $\mathfrak{b}' = \mathfrak{b}^1, \dots, \mathfrak{b}^p = \mathfrak{b}''$ and $\mathfrak{b}' = \mathfrak{b}^1, \dots, \mathfrak{b}^q = \mathfrak{b}'''$ (with corresponding sequences of lines $\ell'^1, \dots, \ell'^{p-1}$ and $\ell''^1, \dots, \ell''^{q-1}$ respectively), so that α and β are simple roots respectively of $\mathfrak{b}''$ and $\mathfrak{b}'''$, and $\mathfrak{g}^{\ell'^i} \simeq sl(1+\varepsilon)$ for $i = 1, \dots, p-1$ and $\mathfrak{g}^{\ell''^i} \simeq sl(1+\varepsilon)$ for $i = 1, \dots, q-1$. Then $V_{\mathfrak{b}}(\nu_\lambda)$ is L_m-integrable iff

$$\begin{aligned} &2(\lambda^{\mathfrak{b}'}_{\mathfrak{b}}, \gamma)/(\gamma, \gamma) \in \mathbb{Z}_+ \quad \text{for every even root } \gamma \text{ of } \mathfrak{b}', \\ &2(\lambda^{\mathfrak{b}''}_{\mathfrak{b}}, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_+, \\ &2(\lambda^{\mathfrak{b}'''}_{\mathfrak{b}}, \beta)/(\beta, \beta) \in \mathbb{Z}_+. \end{aligned}$$

c) Assume that Theorem 2 c) holds.

In this case for every i, $1 \le i \le i_{\mathfrak{k}}$ there exist $\mathfrak{b}^i$ and chain $\mathfrak{b} = \mathfrak{b}^{i,1}, \dots, \mathfrak{b}^{i,k_i} = \mathfrak{b}^i$ (with corresponding sequence of lines $\ell^{i,1}, \dots, \ell^{i,k_i-1}$), so that the basis of $\mathfrak{b}^i \cap \mathfrak{g}^i_0$

contains all simple roots of $\mathfrak{b}^i$ which lie in Θ^i, and $\mathfrak{g}^{\ell^{i,j}} \simeq sl(1+\varepsilon)$ for $j=1,\dots,k_i-1$. Then $V_{\mathfrak{b}}(\nu_\lambda)$ is L_m-integrable iff

$$2(\lambda_{\mathfrak{b}}^{\mathfrak{b}^i},\gamma)/(\gamma,\gamma)\in\mathbb{Z}_+ \text{ for every even root } \gamma \text{ of } \mathfrak{b}^i_{\mathfrak{k}} \text{ which belongs to } \Theta^i.$$

Remark. The explicit conditions on λ, computed in [**KW**], which ensure L_m-integrability (or simply integrability in the terminology of [**KW**]) for $L_m = L^1 \cup L^2 \cup L^{odd}$, $L' = L'_f$, $L^2 = \Theta^2 \cap L^2_f$, Θ^2 being spanned by the roots of $\mathfrak{k}^2$, are nothing but the conditions that the above algorithm produces for this particular set L_m and for the fixed Borel subsuperalgebra considered in [**KW**].

2.2. $\mathfrak{g}=\mathbf{W}_{\mathbf{pol}}(m+n\varepsilon)$. Let $T\simeq\mathbb{C}^{m+n\varepsilon}$ be a $m+n\varepsilon$-dimensional vector space with basis $x_1,\dots,x_m,\xi_1,\dots,\xi_n$ ($x_i\in T_0$, $\xi_j\in T_1$). By definition, $\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon) := \mathrm{der} S^{\cdot}(T)$ (der denoting superderivations). In coordinates, any element of $\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon)$ has the form $\sum_{i=1}^m f_i\frac{\partial}{\partial x_i}+\sum_{j=1}^n g_j\frac{\partial}{\partial \xi_j}$, where $f_i, g_j\in S^{\cdot}(T)$. $\mathfrak{g}=\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon)$ has an obvious $\mathbb{Z}$-filtration $\mathfrak{g}\supset\mathfrak{g}^0\supset\mathfrak{g}^1\supset\cdots\supset\mathfrak{g}^k\supset\cdots$. Namely, $\sum_{i=1}^m f_i\frac{\partial}{\partial x_i}+\sum_{j=1}^n g_i\frac{\partial}{\partial \xi_j}\in\mathfrak{g}^k$ iff $\deg\ f_i\geq k+1$ for every i and $\deg g_j\geq k+1$ for every j. The abelian Lie subsuperalgebra $\langle x_1\frac{\partial}{\partial x_1},\dots,x_m\frac{\partial}{\partial x_m},\xi_1\frac{\partial}{\partial \xi_1},\dots,\xi_n\frac{\partial}{\partial \xi_n}\rangle$ is a Cartan subsuperalgebra of $\mathfrak{g}$. Clearly, as $\mathfrak{b}$-module $\mathfrak{g}$ has a decomposition (1) with finite-dimensional $\mathfrak{h}$-semisimple root spaces $\mathfrak{g}^{(\alpha)}$, and $\Delta=\{\alpha=\sum_{k=1}^m k_i\varepsilon_i+\sum_{j=1}^n l_j\delta_j \mid \alpha\neq 0,\ k_i=-1,0,1,\dots,\ l_j=-1,0,1,\dots,$ and at most one of k_i and l_j is $-1\}$, where ε_i and δ_j are the dual elements respectively of $x_i\frac{\partial}{\partial x_i}$ and $\xi_j\frac{\partial}{\partial \xi_j}$.

The elements $x_p\frac{\partial}{\partial x_i}$, $\xi_q\frac{\partial}{\partial x_i}$, $x_p\frac{\partial}{\partial \xi_j}$, $\xi_q\frac{\partial}{\partial \xi_j}$, where $1\leq i,\ p\leq m$, $1\leq j,\ q\leq n$ generate a finite-dimensional Lie superalgebra $\mathfrak{k}$ of $\mathfrak{g}$ which is isomorphic to $gl(m+n\varepsilon)$; $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ too. We introduce a bilinear form on $\mathfrak{h}^*$ by setting $(\varepsilon_i,\varepsilon_p)=\delta_{ip}$, $(\delta_j,\delta_q)=\delta_{jq}$, $(\varepsilon_i,\delta_j)=0$. If $\alpha=\sum_{i=1}^m k_i\varepsilon_i+\sum_{j=1}^n l_j\delta_j$, then $\mathfrak{g}^{(\alpha)}$ is the (complex) vector space spanned by $\{x_i x_1^{k_1}\cdots x_n^{k_n}\xi_1^{l_1}\cdots\xi_m^{l_m}\frac{\partial}{\partial x_i},\ \xi_j x_1^{k_1}\cdots x_n^{k_n}\cdot \cdot\xi_1^{l_1}\cdots\xi_m^{l_m}\frac{\partial}{\partial \xi_j}\mid i=1,\dots,m,\ j=1,\dots,n\}$.

If E is a $\mathfrak{g}$-module, we say (following A. Rudakov, see [**R**]) that E is of height $p\geq 0$ iff there exists $0\neq v\in E$, such that $\mathfrak{g}^{p+1}\cdot v=0$ but $\mathfrak{g}^p\cdot v'\neq 0$ for every $v'\neq 0$. If E is not of height p for any $p\geq 0$, we say that E is not of finite height. By Γ (resp. Γ^0) we denote the real vector space spanned by Δ (resp. $\varepsilon_1,\dots,\varepsilon_m$). We call a line l of $\mathfrak{g}$ essential iff $\mathfrak{g}^l\cap\mathfrak{h}\neq 0$, and inessential otherwise. The only essential lines of $\mathfrak{g}$ are $\mathbb{R}\varepsilon_i$, $\mathbb{R}(\varepsilon_i-\varepsilon_p)$, $\mathbb{R}\delta_j$, $\mathbb{R}(\delta_j-\delta_q)$.

Let $\mathfrak{b}$ be a Borel subsuperalgebra determined by a real flag $F=\{0\subset F^1\subset\cdots\subset F^{2(m+n)}=\mathfrak{h}_0^*\}$. Denote by Γ_F the hyperplane in Γ obtained by intersecting Γ with F^p for a suitable p. Fix a normal vector a to Γ_F, $a=c_1\varepsilon_1+\cdots+c_m\varepsilon_m+d_1\delta_1+\cdots+d_n\delta_n$, such that $c_i\geq 0$. We shall consider the following alternatives for the m-tuple $(c_1,\dots,c_m)$:

1. $c_i>0$ for $i=1,\dots,m$. In this case it is not hard to check that $\mathfrak{b}$ is determined by a regular real hyperplane P in $\mathfrak{h}_0^*$. Moreover $\mathfrak{n}^+$ or $\mathfrak{n}^-$ is finite-dimensional and $\mathfrak{b}$ has always $m+n$ simple lines. By appropriately choosing the signs of the connected components of $\mathfrak{h}_0^*\backslash P$ (see 1.2) we can assume that $\dim\mathfrak{n}^-<\infty$. The reader will verify immediately that for every λ and for every inessential line l of $\mathfrak{b}$ $\mathfrak{g}^l$ acts

trivially on the $\mathfrak{b}$-highest weight space ν_λ of $V_{\mathfrak{b}}(\nu_\lambda)$.[7)] Furthermore we have

PROPOSITION 4. *Assume that $c_i > 0$ for $i = 1, \ldots, m$ and let $\lambda \in \mathfrak{h}^*$. Then*

a) *The $\mathfrak{k}$-module structure of $V_{\mathfrak{k}\cap\mathfrak{b}}(\nu_\lambda)$ can be uniquely extended to the structure of a $(\mathfrak{b}+\mathfrak{k})$-module with $\mathfrak{b}$-highest weight space ν_λ. Therefore $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}+\mathfrak{k})} V_{\mathfrak{k}\cap\mathfrak{b}}(\nu_\lambda)$ is a $\mathfrak{b}$-highest weight $\mathfrak{g}$-module and the canonical projection $\tilde{V}_{\mathfrak{b}}(\nu_\lambda) \to V_{\mathfrak{b}}(\nu_\lambda)$ factors through $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}+\mathfrak{k})} V_{\mathfrak{k}\cap\mathfrak{b}}(\nu_\lambda)$, i.e. one has a natural commutative diagram:*

$$\begin{array}{ccc} \tilde{V}_{\mathfrak{b}}(\nu_\lambda) & \longrightarrow & V_{\mathfrak{b}}(\nu_\lambda) \\ & \searrow \qquad\qquad \nearrow & \\ & U(\mathfrak{g}) \otimes_{U(\mathfrak{b}+\mathfrak{k})} V_{\mathfrak{k}\cap\mathfrak{b}}(\nu_\lambda). & \end{array}$$

b) *$V_{\mathfrak{b}}(\nu_\lambda)$ is of height 1 for every λ.*

c) *$V_{\mathfrak{b}}(\nu_\lambda)$ is L_f-integrable iff $V_{\mathfrak{b}\cap\mathfrak{k}}(\lambda^\delta)$ is finite-dimensional.*

SKETCH OF PROOF. There exists a chain $\mathfrak{b} = \mathfrak{b}^1, \ldots, \mathfrak{b}^k = \mathfrak{b}'$, such that the corresponding sequence of lines $\ell^1, \ldots, \ell^{k-1}$ consists of inessential lines only and $\mathfrak{b}'$ can be determined by a flag F' for which $\Delta_{\mathfrak{k}} \subset \Gamma_{F'}$. Then $V_{\mathfrak{b}}(\nu_\lambda) \simeq V_{\mathfrak{b}'}(\nu_\lambda)$ and the corresponding statements for $V_{\mathfrak{b}'}(\nu_\lambda)$ follow easily from the fact that $\Gamma_{F'}$ contains all essential finite lines of $\mathfrak{g}$. $\square$

2. At least one of $c_1, \ldots, c_m$ is non-positive. In this case the subspace $\Gamma_F \cap \Gamma^0$ of Γ^0 is uniquely determined by $\mathfrak{b}$. This implies in particular that the set of all Borel subsuperalgebras of $\mathfrak{g}$ is uncountable. The length of the shortest flag which determines $\mathfrak{b}$ can vary from 1 to m. Moreover $\mathfrak{b}$ has less than $m+n$ simple lines and the height of $V_{\mathfrak{b}}(\nu_\lambda)$ is not finite unless $\lambda = 0$. Finally, one can check that for such a $\mathfrak{b}$ L_f is never contained in a $\leq^{\mathcal{O}_{\mathfrak{b}}^{\mathfrak{C}}}$-maximal set of lines.

Around 1980 J. Bernstein and D. Leites initiated the study of representations of the Cartan type Lie superalgebras, [**BL**]. They relied on the pioneering paper of A. Rudakov [**R**]. A novelty of our approach is that we single out the highest weight modules. Bernstein and Leites did not consider modules whose height is not finite and it seems now that a study of more general classes of representations of the Cartan series of Lie superalgebras is desirable.

3. Loop modules

In this section $\mathfrak{g} = \mathfrak{k}^{loop}$, $\hat{\mathfrak{k}}$, where $\mathfrak{k}$ denotes one of the Lie superalgebras $sl(m+n\varepsilon)$, $m+n\varepsilon > 1+\varepsilon$, $osp(m+2n\varepsilon)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$.

As we pointed out in the introduction there are $\mathfrak{g}$-modules in the category $\mathfrak{C}$ which are L_f-integrable but which are not highest weight modules. The most obvious example of such a module is $\mathfrak{g}$ itself, i.e. the adjoint module. We will now construct a large class of such $\mathfrak{g}$-modules.

If $\mathfrak{b}_{\mathfrak{k}}$ is a Borel subsuperalgebra in $\mathfrak{k}$, $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{n}_{\mathfrak{k}}^+$, we say that a $\mathfrak{g}$-module V from the category $\mathfrak{C}$ is $\mathfrak{b}_{\mathfrak{k}}$-*bounded*, if it is generated by a generalized weight vector

[7)] For $\mathbf{W}_{\mathrm{pol}}(m+n\varepsilon)$, $\dim\nu_\lambda$ equals 1 or ε since here $\mathfrak{h} = \mathfrak{h}_0$.

v on which $\mathfrak{n}^+_{\mathfrak{k}} \otimes \mathbb{C}[t, t^{-1}]$ acts trivially. $\mathfrak{C}_{\mathfrak{b}_{\mathfrak{k}}}$ denotes the subcategory of $\mathfrak{g}$-modules which are generated by finitely many $\mathfrak{b}_{\mathfrak{k}}$-bounded $\mathfrak{g}$-submodules. Fixing $\mathfrak{b}_{\mathfrak{k}}$, we set

$$N^+ = \mathfrak{n}^+_{\mathfrak{k}} \otimes \mathbb{C}[t, t^{-1}], \quad N^- = \mathfrak{n}^-_{\mathfrak{k}} \otimes \mathbb{C}[t, t^{-1}],$$
$$H = \mathfrak{h} + (\mathfrak{h}_{\mathfrak{k}} \otimes \mathbb{C}[t, t^{-1}]), \quad T = \bigoplus_{n \in \mathbb{Z}\setminus\{0\}} (\mathfrak{h}_{\mathfrak{k}} \otimes t^n).$$

Define the $H \ni\!\!\!\supset N^+$-module $\bar{\nu}$ as $\bar{\nu} := U(H \ni\!\!\!\supset N^+) \otimes_{U(N^+)} \mathbb{C} \cdot v$. Then $\bar{\nu}$ has a unique irreducible submodule $\nu(\Lambda)$, where $\Lambda : U(T) \to \mathbb{C}[t, t^{-1}]$ is a graded homomorphism with $\Lambda(1) = 1$. Λ is determined by a sequence of linear functions $c_s \in \mathfrak{h}^*_{\mathfrak{k}}$ [8)], $s \in \mathbb{Z}\setminus\{0\}$, via $\Lambda(h \otimes t^s) = c_s(h)t^s$ for every $h \in \mathfrak{h}_{\mathfrak{k}}$. Set $\bar{V}(\Lambda) = U(\mathfrak{g}) \otimes_{U(H \ni\!\!\!\supset N^+)} \nu(\Lambda)$. Obviously $\bar{V}(\Lambda)$ is $\mathfrak{h}$-semisimple (with infinite-dimensional weight spaces), and an argument very similar to the one in the case of highest weight modules proves that $\bar{V}(\Lambda)$ has a unique irreducible submodule $V(\Lambda)$ (see [**C**]). Every irreducible object of $\mathfrak{C}_{\mathfrak{b}_{\mathfrak{k}}}$ is isomorphic to $V(\Lambda)$ for some Λ.

Theorem 3. *$V(\Lambda)$ is a L_f-integrable module with finite-dimensional ($\mathfrak{h}$-semisimple) weight spaces iff there exist finitely many weights $\lambda_i \in \mathfrak{h}^*_{\mathfrak{k}}$, $i = 1, \ldots, N$, such that* $\dim V_{\mathfrak{b}_{\mathfrak{k}}}(\nu_{\lambda^i}) < \infty$ *and such that for every $h \in \mathfrak{h}_{\mathfrak{k}}$ and every $s \in \mathbb{Z}\setminus\{0\}$*

$$\Lambda(h \otimes t^s) = \left(\sum_{i=1}^{N} \lambda_i(h) \xi_i^s \right) t^s, \tag{11}$$

ξ_i being certain non-zero complex constants.

We will present the proof of Theorem 3 in a forthcoming publication. Here we shall restrict ourselves to a few remarks. Theorem 3 is a partial generalization of the main result of [**C**]. Condition (11) is the loop module analog of the integrality condition for highest weight modules and it means roughly that for every fixed h the sequence $c_s(h)$ satisfies a recurrent equation. The main new ingredient needed to prove Theorem 3 is the study of bounded modules over the Lie superalgebra $gl(1+\varepsilon)^{loop} := gl(1+\varepsilon) \otimes \mathbb{C}[t, t^{-1}] \in\!\!\!\!\supset \mathbb{C}D$. There are interesting effects arising here. One of them is that there are L_f-integrable irreducible $gl(1+\varepsilon)^{loop}$-modules which are $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces but are not bounded. This is related to why the following problem is still open: is it true that any irreducible L_f-integrable $\mathfrak{g}$-module which is $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces is bounded?

Following V. Chari and A. Pressley, [**CP**], we introduce a loop module as an irreducible component of a $\mathfrak{g}$-module of the form

$$V_{\mathfrak{b}_{\mathfrak{k}}}(\nu_{\lambda^1}) \otimes \cdot \ldots \cdot \otimes V_{\mathfrak{b}_{\mathfrak{k}}}(\nu_{\lambda^i}) \otimes \mathbb{C}[t, t^{-1}],$$

$V_{\mathfrak{b}_{\mathfrak{k}}}(\nu_{\lambda^j})$ being irreducible $\mathfrak{b}_{\mathfrak{k}}$-highest weight $\mathfrak{k}$-modules. Theorem 3 implies

Corollary 2. *Every L_f-integrable $\mathfrak{b}_{\mathfrak{k}}$-bounded $\mathfrak{g}$-module V which is $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces is a loop module.* $\square$

Another effect worth to be noticed is that Corollary 2 is not true for $gl(1+\varepsilon)^{loop}$. We conclude this paper by two more Corollaries of Theorem 3.

8) In the cases considered $\mathfrak{h}_{\mathfrak{k}} = (\mathfrak{h}_{\mathfrak{k}})_0$.

Corollary 3. *An irreducible $\mathfrak{b}_\mathfrak{k}$-bounded $\mathfrak{g}$-module V with finite-dimensional weight spaces is L_f-integrable iff V is $\mathfrak{b}'_\mathfrak{k}$-bounded for every Borel subsuperalgebra $\mathfrak{b}'_\mathfrak{k}$ of $\mathfrak{k}$ with $\mathfrak{b}'_\mathfrak{k} \supset \mathfrak{h}_\mathfrak{k}$.* $\square$

Corollary 4. *For any $\mathfrak{b}_\mathfrak{k}$, L_f is the only maximal set of lines with respect to $\leq^{\mathfrak{C}_{\mathfrak{b}_\mathfrak{k}}}$.* $\square$

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