PARTIALLY INTEGRABLE HIGHEST WEIGHT MODULES

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Abstract. We prove a more general version of a result announced without proof in [DP], claiming roughly that in a partially integrable highest weight module over a Kac-Moody algebra the integrable directions form a parabolic subalgebra.

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Introduction

Integrable highest weight modules of Kac-Moody algebras are analogues of finite-dimensional modules and have been studied in detail, see [K2]. Based on the work of Kac and Wakimoto, [KW], in the earlier paper [DP] we initiated the study of partially integrable highest weight modules. In particular we announced the statement (Proposition 3, [DP]) that in a partially integrable highest weight module of an affine Lie algebra the integrable directions form a parabolic subalgebra. The main point of the present paper is to prove a more general theorem for partially integrable modules over Lie superalgebras. Another way of looking at this theorem is as an infinite-dimensional version of a result in [DMP].

Here is a brief summary of the paper. As we are dealing with highest weight modules, we take the opportunity to introduce a rather general definition of a Borel subsuperalgebra of an arbitrary Lie superalgebra with generalized root decomposition. It enables us to consider highest weight modules in natural generality. We then introduce the class of weak Kac-Moody superalgebras which contains in particular usual Kac-Moody algebras, affine Lie superalgebras (with or without central extension) and direct limit Lie algebras (and Lie superalgebras) like $A(\infty)$. In section 2 we establish the main result which gives a complete characterization of integrable directions in certain highest weight modules of weak Kac-Moody superalgebras. The paper is concluded by a detailed discussion of examples.

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1. Notations and preliminaries

1.0. **Notations.** The ground field is $\mathbb{C}$. The signs $\Subset$ and $\Supset$ stand for semi-direct sum of Lie superalgebras. All vector spaces are automatically $\mathbb{Z}_2$-graded, but when Lie algebras (and their representations) are considered it is assumed that the $\mathbb{Z}_2$-grading is trivial, i.e. that the odd part is zero. For a vector space, the subscripts $_0$ and $_1$ refer always to the $\mathbb{Z}_2$-grading; the dimension of a vector space is written as

$m+n\varepsilon$, ε denoting a formal odd variable with $\varepsilon^2=1$. The superscript * stands for dual space. Complex (respectively, real) span is denoted by $<\ \ >_{\mathbb{C}}$ (resp. $<\ \ >_{\mathbb{R}}$), δ^{ij} is Kronecker's delta, $\mathbb{R}_+ := \{r\in\mathbb{R}\,|\,r\geq 0\}$, $\mathbb{R}_- := \{r\in\mathbb{R}\,|\,r\leq 0\}$ and $\mathbb{Z}_+ := \mathbb{Z}\cap\mathbb{R}_+$.

1.1. **Locally finite actions and integrable modules.** If $\mathfrak{g}$ is any Lie superalgebra, V is a $\mathfrak{g}$-module and $g\in\mathfrak{g}$ is any element in $\mathfrak{g}$, we say that g *acts locally finitely on* V if $\dim(<\{g^r\cdot \mathrm{v}\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})<\infty$ for any $\mathrm{v}\in V$.

Lemma 1. *If V is a $\mathfrak{g}$-module generated by a single vector v^0 and g acts locally finitely on $\mathfrak{g}$ via the adjoint representation, then g acts locally finitely on V iff g acts finitely on v^0, i.e. iff* $\dim(<\{g^r\cdot \mathrm{v}^0\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})<\infty$.

Proof. Clearly we only need to show that if g acts finitely on v^0, then g acts locally finitely on V. Using Jacobi's identity, the reader will verify the inequality

$$\begin{aligned}&\dim(<\{g^r\cdot x\cdot \mathrm{v}^0\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})\leq\\ &\quad\dim(<\{(\mathrm{ad}g)^r(x)\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})\ \dim(<\{g^r\cdot \mathrm{v}^0\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})\end{aligned}\tag{1}$$

for any $x\in\mathfrak{g}$. Applying (1) s times with v^0 consequtively replaced by the vectors $\mathrm{v}^0, x^1\cdot\mathrm{v}^0,\dots,x^{s-1}\cdots x^1\cdot\mathrm{v}^0$, one concludes that $\dim(<\{g^r\cdot\mathrm{v}'\,|\,r\in\mathbb{Z}_+\}>_{\mathbb{C}})<\infty$ for any v' of the form $x^s\cdots x^1\cdot\mathrm{v}^0$ with arbitrary $x^1,\dots,x^s\in\mathfrak{g}$. But since V is generated by v^0, any vector in V is a finite sum of vectors v', and thus Lemma 1 is proved. □

A $\mathfrak{g}$-module V is *integrable* iff any $g\in\mathfrak{g}$ acts locally finitely on V.

1.2. **Lie superalgebras with root decomposition and generalized weight modules.** Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a Lie superalgebra. A *Cartan subsuperalgebra of* $\mathfrak{g}$ is by definition a self-normalizing nilpotent Lie subsuperalgebra $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\subset\mathfrak{g}$. We do not assume $\mathfrak{g}$ or $\mathfrak{h}$ to be finite-dimensional. A $\mathfrak{g}$-module V is a *generalized weight $\mathfrak{g}$-module* iff V is integrable as an $\mathfrak{h}_0$-module. Since the $\mathfrak{h}_0$-module $U(\mathfrak{h}_0)\cdot\mathrm{v}$ is finite-dimensional for any $\mathrm{v}\in V$ and since every finite-dimensional $\mathfrak{h}_0$ module has 1- or ε-dimensional composition factors, as an $\mathfrak{h}_0$-module V decomposes as the direct sum $\oplus_{\lambda\in\mathfrak{h}_0^*}V^\lambda$, where

$$V^\lambda := \{\mathrm{v}\in V\,|\,\text{for every } h\in\mathfrak{h}_0, h-\lambda(h)\,\text{acts nilpotently on v}\}$$

We call V^λ the *generalized weight space of V of weight* λ. Note that if $V^\lambda\neq 0$, one necessarily has $\lambda|_{[\mathfrak{h}_0,\mathfrak{h}_0]}=0$. We define a linear function $\lambda\in\mathfrak{h}_0^*$ to be a *weight* iff $\lambda|_{[\mathfrak{h}_0,\mathfrak{h}_0]}=0$. Note also that each generalized weight space V^λ is automatically an $\mathfrak{h}$-module.

Henceforth $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ will denote a Lie superalgebra with a fixed proper Cartan subsuperalgebra $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1$ such that $\mathfrak{g}$ is a generalized weight module, i.e.

$$\mathfrak{g}=\mathfrak{h}\oplus(\oplus_{\alpha\in\mathfrak{h}_0^*\setminus\{0\}}\mathfrak{g}^\alpha).\tag{2}$$

The generalized weight spaces $\mathfrak{g}^\alpha$ are by definition the *root spaces* of $\mathfrak{g}$, $\Delta:=\{\alpha\in\mathfrak{h}_0^*\setminus\{0\}|\mathfrak{g}^\alpha\neq 0\}$ is the set of *roots of* $\mathfrak{g}$, and the decomposition (2) is the *root decomposition of* $\mathfrak{g}$. All $\mathfrak{g}$-modules V we consider below will be assumed to be generalized weight modules. The *support of* V, $\mathrm{supp}V$, consists of all weights λ with $V^\lambda\neq 0$.

1.3. **Lines and line subsuperalgebras.** We call a *line of* $\mathfrak{g}$ any 1-dimensional real subspace ℓ of $\mathbb{R}\Delta := <\Delta>_{\mathbb{R}}$ whose intersection with the set of roots Δ of $\mathfrak{g}$ is non-empty. Given a line ℓ of $\mathfrak{g}$, its *line Lie subsuperalgebra* $\mathfrak{g}^\ell$, or *line subsuperalgebra* for short, is the Lie subsuperalgebra of $\mathfrak{g}$ generated by all root spaces $\mathfrak{g}^\alpha$ for $\alpha \in \ell \cap \Delta$. A line ℓ is *finite* iff $\dim \mathfrak{g}^\ell < \infty$, and is *infinite* otherwise. Proposition 3 in [PS] implies that for a finite line ℓ there are only the following alternatives:

(i) $\mathfrak{g}^\ell$ is a nilpotent Lie superalgebra,
(ii) $\mathfrak{g}^\ell \simeq \mathfrak{r} \Subset sl(2)$,
(iii) $\mathfrak{g}^\ell \simeq \mathfrak{r} \Subset osp(1+2\varepsilon)$,

where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}^\ell$ and moreover this radical is nilpotent. In particular, if $\mathfrak{g}^\ell$ is not nilpotent, the semi-simple part of the Lie algebra $\mathfrak{g}^\ell_{\bar{0}}$ is isomorphic to $sl(2)$ and there is a *standard basis* e_ℓ, h_ℓ, f_ℓ of $\mathfrak{g}^\ell_{\bar{0}}$ with $[e_\ell, f_\ell] = h_\ell, [h_\ell, \ell_\ell] = \alpha_\ell(h_\ell)e_\ell, [h_\ell, f_\ell] = -\alpha_\ell(h_\ell)f_\ell$ for some fixed root $\alpha_\ell \in \ell$. In what follows we will call a finite line ℓ of $\mathfrak{g}$

- *nilpotent* if $\mathfrak{g}^\ell$ is nilpotent;
- an $sl(2)$-*line* if $\mathfrak{g}^\ell$ is not nilpotent.

1.4. **Borel subsuperalgebras.** A decomposition

$$\Delta = \Delta^+ \sqcup \Delta^- \tag{3}$$

is a *triangular decomposition of* Δ iff the cone $\mathbb{R}_+(\Delta^+ \cup -\Delta^-)$ [1] (or equivalently, its opposite cone $\mathbb{R}_+(-\Delta^+ \cup \Delta^-)$) contains no (real) vector subspace. Equivalently, (3) is a triangular decomposition if the following is a well-defined $\mathbb{R}$-linear partial order on $\mathbb{R}\Delta$:

$$\eta \geq \mu \Leftrightarrow \begin{array}{l} \eta = \mu + \sum_{i=1}^n c^i\alpha^i \text{ for some } \alpha^i \in \Delta^+ \cup -\Delta^- \text{ and some } c^i \in \mathbb{R}_+, \\ \text{or } \mu = \eta. \end{array} \tag{4}$$

(A partial order is $\mathbb{R}$-linear if it is compatible with addition and multiplication by $\mathbb{R}_+$ and if multiplication by $\mathbb{R}_-$ changes the order direction).

Any regular real hyperplane H in $\mathbb{R}\Delta$ (i.e. a codimension 1 subspace H with $H \cap \Delta = \emptyset$) together with a labeling of the two connected components of $\mathbb{R}\Delta \backslash H$ as $(\mathbb{R}\Delta\backslash H)^+$ and $(\mathbb{R}\Delta\backslash H)^-$ determines a triangular decomposition of Δ:

$$\Delta^\pm := \Delta \cap (\mathbb{R}\Delta\backslash H)^\pm.$$

If Δ is finite, then conversely, every triangular decomposition is obtained in this way. If Δ is infinite, the process of constructing a triangular decomposition from a hyperplane can be generalized for instance as follows. A flag $F = \{\ldots \subset F^i \subset \ldots \subset F^{i+r} \subset \ldots \subset \mathbb{R}\Delta\}$ of linear subspaces in $\mathbb{R}\Delta$ is a *full flag* iff it admits no refinement, i.e. iff $\dim(F^i/F^{i-1}) = 1$ for all $i \geq 1$. We define F to be *regular* iff $\bigcup_i F^i = \mathbb{R}\Delta$ and $(\bigcap_i F^i) \bigcap \Delta = \emptyset$. Finally we call F *oriented* if for every i the two connected components of $F^{i+1}\backslash F^i$ are labeled (in an arbitrary way) as $(F^{i+1}\backslash F^i)^+$ and $(F^{i+1}\backslash F^i)^-$. Every regular oriented full flag F in $\mathbb{R}\Delta$ determines now a triangular decomposition of Δ:

$$\Delta^\pm := \sqcup_i (\Delta^i)^\pm,$$

[1] In the present paper, cone is a synonym for an $\mathbb{R}_+$-invariant additive subset of a vector space.

where $(\Delta^i)^\pm := \Delta \cap (F^{i+1}\backslash F^i)^\pm$. F determines also an $\mathbb{R}$-linear partial lexicographical order on $\mathbb{R}\Delta\backslash(\cap_k F^k)$. In terms of this latter order, the $\mathbb{R}$-linear partial order on $\mathbb{R}\Delta$ can be written as

$$\eta \geq \mu \Leftrightarrow \eta = \mu + \beta, \beta \in \mathbb{R}\Delta, \text{ for some } \beta > 0, \text{ or } \mu = \eta,$$

where $\beta > 0$ iff $\beta \in (F^{i+1}\backslash F^i)^+$, i being the only index for which $\beta \in F^{i+1}\backslash F^i$.

If $\dim\mathbb{R}\Delta < \infty$, the reader will verify that fixing an $\mathbb{R}$-linear order on $\mathbb{R}\Delta$ is equivalent to fixing a regular oriented full flag F (i.e. an oriented flag of length $\dim\mathbb{R}\Delta + 1$) in $\mathbb{R}\Delta$, and moreover that each triangular decomposition of Δ is determined by some (in general not unique) regular oriented full flag, cf. [DP].

A Lie subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is by definition a *Borel subsuperalgebra of* $\mathfrak{g}$ iff $\mathfrak{b} = \mathfrak{h} \ni (\oplus_{\alpha\in\Delta^+}\mathfrak{g}^{(\alpha)})$ for some triangular decomposition $\Delta = \Delta^+ \sqcup \Delta^-$. Adopting terminology from affine Kac-Moody algebras, we will call a Borel subsuperalgebra *standard* iff it corresponds to a triangular decomposition which can be determined by a regular hyperplane H in $\mathbb{R}\Delta$. When $\dim\mathfrak{g} < \infty$, every Borel subsuperalgebra is standard. The partial order (4) corresponding to a given Borel subsuperalgebra $\mathfrak{b}$ will be denoted from now on by $\geq_\mathfrak{b}$.

In the proof of our main Theorem (in section 2 below) we will need the notion of a chain of Borel subsuperalgebras. First we define two non-equal Borel subsuperalgebras $\mathfrak{b}'$ and $\mathfrak{b}''$ of $\mathfrak{g}$ to be *adjacent* , cf. [PS], iff they correspond to triangular decompositions $\Delta = (\Delta^+)' \sqcup (\Delta^-)'$ and $\Delta = (\Delta^+)'' \sqcup (\Delta^-)''$ such that

$$(\Delta^+)'\backslash((\Delta^+)' \cap \ell) = (\Delta^+)''\backslash((\Delta^+)'' \cap \ell) \tag{5}$$

for some (unique) line ℓ of $\mathfrak{g}$. A line ℓ of $\mathfrak{g}$ is *simple for a Borel subsuperalgebra* $\mathfrak{b}'$ if there exists a Borel subsuperalgebra $\mathfrak{b}''$ of $\mathfrak{g}$ so that (5) holds. Finally, a sequence of Borel subsuperalgebras

$$\dots, \mathfrak{b}^i, \dots, \mathfrak{b}^{i+k}, \dots$$

is a *chain* iff $\mathfrak{b}^j$ and $\mathfrak{b}^{j+1}$ are adjacent for all j. Note that every chain determines a sequence of lines of $\mathfrak{g}$

$$\dots, \ell^i, \dots, \ell^{i+k-1}, \dots \quad ,$$

where ℓ^j is determined uniquely by the pair $\mathfrak{b}^{j-1}, \mathfrak{b}^j$ of adjacent Borel subsuperalgebras.

1.5. **Highest weight modules.** Let us observe first that if v is an irreducible generalized weight $\mathfrak{b}$-module (the Cartan subsuperalgebra of $\mathfrak{b}$ is $\mathfrak{h}$), then $\mathfrak{n}^+ := \oplus_{\alpha\in\Delta^+}\mathfrak{g}^\alpha$ necessarily acts trivially on v (since otherwise $U(\mathfrak{n}^+)\cdot v$ would be a proper $\mathfrak{b}$-submodule) and thus v is simply an irreducible generalized weight $\mathfrak{h}$-module. If $\dim\mathfrak{h}_1 < \infty$, v is finite-dimensional (v necessarily has a weight vector v, i.e. a vector v on which $\mathfrak{h}_0$ acts via a weight λ, and therefore v is a quotient of the finite-dimensional $\mathfrak{h}$-module $U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_0)} \mathbb{C}\mathrm{v}$). Moreover, v consists of a unique generalized weight space and is determined up to parity change by its weight. An explicit description of all irreducible finite-dimensional modules over a finite-dimensional solvable Lie superalgebra has been given by Kac in [K1] (see also [PS]). If $\dim\mathfrak{h}_1 = \infty$, v may be infinite-dimensional but it is still necessarily a quotient of $U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_0)} \mathbb{C}\mathrm{v}$ for some weight vector v, and therefore v always has a unique generalized weight space. In what follows we will always denote an irreducible generalized weight $\mathfrak{h}$-(or $\mathfrak{b}$-)module by $v_\lambda, \lambda \in \mathfrak{h}_0^*$ being its respective weight.

A $\mathfrak{b}$-*highest weight module* is a generalized weight $\mathfrak{g}$-module V whose support $\mathrm{supp}V$ belongs to a shift of the cone $\mathbb{R}_+(\Delta^+ \cup -\Delta^-)$ by a weight in $\mathfrak{h}_0^*$, and which is generated by a weight space V^λ such that λ is a maximal element of supp$V$ with respect to $\geq_{\mathfrak{b}}$ and V^λ is an irreducible $\mathfrak{h}$-module. The weight space V^λ, which is obviously unique, is by definition the *highest weight space of* V. Furthermore, V^λ is necessarily an irreducible $\mathfrak{b}$-submodule of V. Conversely, if v_λ is any irreducible $\mathfrak{b}$-module, then the *Verma module*

$$\tilde{V_{\mathfrak{b}}}(v_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} v_\lambda$$

is a $\mathfrak{b}$-highest weight module, and moreover any $\mathfrak{b}$-highest weight module with highest weight space v_λ is isomorphic to a quotient of $\tilde{V_{\mathfrak{b}}}(v_\lambda)$. Indeed, note first that every $\mathfrak{g}$-submodule of $\tilde{V_{\mathfrak{b}}}(v_\lambda)$ is a generalized weight module (more generally, a non-difficult extension of the argument in the proof of Proposition 1.5 in [K2] shows that every $\mathfrak{g}$-submodule of a generalized weight $\mathfrak{g}$-module is a generalized weight module), and thus every quotient of $\tilde{V_{\mathfrak{b}}}(v_\lambda)$ is a generalized weight module. Being generated by v_λ, every quotient of $\tilde{V_{\mathfrak{b}}}(v_\lambda)$ is thus a $\mathfrak{b}$-highest weight module. Conversely, given a $\mathfrak{b}$-highest weight module V with highest weight space v_λ, there is an obvious $\mathfrak{g}$-surjection $\tilde{V_{\mathfrak{b}}}(v_\lambda) \to V$. Note finally that $\tilde{V_{\mathfrak{b}}}(v_\lambda)$ has a unique proper maximal $\mathfrak{g}$-submodule (since any element in the support of the sum of all proper submodules in $\tilde{V_{\mathfrak{b}}}(v_\lambda)$ is strictly less than λ with respect to $\geq_{\mathfrak{b}}$) and thus also a unique proper irreducible quotient $V_{\mathfrak{b}}(v_\lambda)$. The latter is by definition the *irreducible* $\mathfrak{b}$*-highest weight module with highest weight space* v_λ.

If ℓ is a line of $\mathfrak{g}$, we call a $\mathfrak{g}$-module V ℓ*-integrable* iff V is an integrable $\mathfrak{g}^\ell$-module. If V is a $\mathfrak{b}$-highest weight module with highest weight space v_λ and the adjoint module $\mathfrak{g}$ is ℓ-integrable, Lemma 1 implies that V is ℓ-integrable iff $\mathfrak{g}^\ell$ acts locally finitely on v_λ. In the proof of our main Theorem below we will also need the following:

Lemma 2. *Let* $\mathfrak{b}$ *be a Borel subsuperalgebra of* $\mathfrak{g}$ *and* α *be a root of* $\mathfrak{b}$ *such that* $\ell = \mathbb{R}\alpha$ *is an* $sl(2)$*-line for which* $\mathfrak{g}$ *is* ℓ*-integrable. Then a* $\mathfrak{b}$*-highest weight module* V *with highest weight space* v_λ *is* ℓ*-integrable iff* f_ℓ *(see* 1.2*) acts locally nilpotently on* v_λ*.*

Proof. As we already noted, Lemma 1 implies that V is ℓ-integrable iff $\mathfrak{g}^\ell$ acts locally finitely on v_λ. The latter requirement is obviously equivalent to the finite-dimensionality of $U(\mathfrak{g}^\ell)\cdot \mathrm{v}$ for any $\mathrm{v} \in v_\lambda$. We claim now that $\dim U(\mathfrak{g}^\ell)\cdot \mathrm{v} < \infty$ iff f_ℓ acts nilpotently on v. This is established as follows. Denote by $(\mathfrak{g}_0^\ell)^{ss}$ the semi-simple part of $\mathfrak{g}_0^\ell$ and by $\mathfrak{r}$ the radical of $\mathfrak{g}$. Consider the surjection of $U((\mathfrak{g}_0^\ell)^{ss})$-modules

$$U(\mathfrak{r}^\ell)\otimes_{U((\mathfrak{g}_0^\ell)^{ss})} < f_\ell^k \cdot \mathrm{v} >_{\mathbb{C}} \longrightarrow U(\mathfrak{g}^\ell)\cdot \mathrm{v},$$

where k runs over $\mathbb{Z}_+$. Since $U(\mathfrak{r}^\ell)$ is an integrable $(\mathfrak{g}_0^\ell)^{ss}$-module, $U(\mathfrak{g}^\ell)\cdot \mathrm{v}$ is integrable as a $(\mathfrak{g}_0^\ell)^{ss}$-module iff f_ℓ acts nilpotently on v. But in the latter case $U(\mathfrak{g}^\ell)\cdot \mathrm{v}$ is necessarily finite-dimensional since it is a $\mathfrak{b}\cap\mathfrak{g}^\ell$-highest weight $\mathfrak{g}^\ell$-module. □

1.6. **Weak Kac-Moody superalgebras.** In order to be able to state our main result in its natural generality we need to introduce the class of weak Kac-Moody superalgebras. We define a Lie superalgebra $\mathfrak{g}$ to be a *weak Kac-Moody superalgebra* [2] iff $\mathfrak{g}$ is ℓ-integrable for every $sl(2)$-line ℓ of $\mathfrak{g}$ and if there is a Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which admits a set $\Sigma_{\mathfrak{b}} \subset \Delta^+$, called *weak basis of* $\mathfrak{b}$, such that:

[2] If $\mathfrak{g}$ is a Lie algebra, we will use the term *weak Kac-Moody algebra.*

- for every $\beta \in \Sigma_{\mathfrak{b}}$, $\mathbb{R}\beta$ is an $sl(2)$-line;
- assuming that $\alpha_\ell \in \Delta^+$, for every $sl(2)$-line ℓ of $\mathfrak{g}$ one has $\alpha_\ell = \sum_{\beta^i \in \Sigma_{\mathfrak{b}}} c^i \beta^i$ for some non-negative constants c^i, and e_ℓ, f_ℓ and h_ℓ belong to the Lie subalgebra $\mathfrak{l}'_\ell$ of $\mathfrak{g}_0$ generated by $e_{\mathbb{R}\beta^i}, f_{\mathbb{R}\beta^i}$ for $\beta^i \in \Sigma_{\mathfrak{b}}$ with $c^i \neq 0$;
- for every $sl(2)$-line ℓ the matrix $A = (\frac{2\alpha_{\ell^p}(h_{\ell^q})}{\alpha_{\ell^p}(h_{\ell^p})})$, where $\ell^p = \ell_{\mathbb{R}\beta^p}$, $\ell^q = \ell_{\mathbb{R}\beta^q}$, and p, q run over the set of indices $\{i | c^i \neq 0\}$, is a generalized Cartan matrix, see [K2], and moreover there exists a Lie subalgebra $\mathfrak{l}_\ell$ of $\mathfrak{g}_0$ with $\mathfrak{l}'_\ell \subset \mathfrak{l}_\ell \subset \mathfrak{l}'_\ell + \mathfrak{h}_0$ which admits an isomorphism of Lie algebras
$$\mathfrak{l}_\ell \simeq \mathfrak{g}(A)/\mathfrak{i},$$
$\mathfrak{g}(A)$ being the Kac-Moody algebra with generalized Cartan matrix A and $\mathfrak{i}$ being an ideal of $\mathfrak{g}(A)$ contained in the center of $\mathfrak{g}(A)$.

If $\mathfrak{g}$ is a weak Kac-Moody superalgebra, a Borel subsuperalgebra which admits a weak basis is by no means unique. Note also that $\mathfrak{g}$ is a weak Kac-Moody superalgebra iff $\mathfrak{g}_0$ is a weak Kac-Moody algebra, and a Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ admits a weak basis iff the Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$ admits a weak basis. Various examples of weak Kac-Moody superalgebras are discussed in section 3 below. Any generalized Kac-Moody algebra with at least one imaginary simple root, see [Bo], provides an example of a Lie algebra which is not a weak Kac-Moody algebra (i.e. no Borel subalgebra which contains the fixed Cartan subalgebra admits a weak basis).

2. The main result

Let $\mathfrak{g}$ be a weak Kac-Moody superalgebra, $\mathfrak{b}$ be a Borel subsuperalgebra with weak basis $\Sigma_{\mathfrak{b}}$, and V be a $\mathfrak{b}$-highest weight $\mathfrak{g}$-module with highest weight space v_λ. By Δ^F_V we denote the set of all roots $\alpha \in \Delta$ such that V is $\mathbb{R}\alpha$-integrable. The following Theorem characterizes all $sl(2)$-lines ℓ for which V is ℓ-integrable.

Theorem. *For an $sl(2)$-line ℓ one has $\alpha_\ell \in \Delta^F_V$ iff $\alpha_\ell \in <\Sigma^V_{\mathfrak{b}}>_{\mathbb{R}}$, where $\Sigma^V_{\mathfrak{b}} := \Sigma_{\mathfrak{b}} \cap \Delta^F_V$.*

Proof. Fix the Lie algebra $\mathfrak{l}_\ell$, the corresponding elements $e_{\ell^i}, h_{\ell^i}, f_{\ell^i} \in \mathfrak{l}_\ell$, the decomposition $\alpha_\ell = \sum_i c^i \beta^i$ and the Kac-Moody algebra $\mathfrak{g}' := \mathfrak{g}(A)$ as in the definition of a weak Kac-Moody superalgebra. According to Lemma 2, f_ℓ acts locally nilpotently on v_λ iff $\alpha_\ell \in \Delta^F_V$. To prove the Theorem we will show that f_ℓ acts locally nilpotently on v_λ iff $\alpha_\ell \in <\Sigma^V_{\mathfrak{b}}>_{\mathbb{R}}$.

As a first step we will reduce the proof of the latter statement to the proof of an analogous statement about any $\mathfrak{b}'$-highest weight $\mathfrak{g}'$-module, where $\mathfrak{b}'$ is the Borel subalgebra of $\mathfrak{g}'$ which projects onto the Borel subalgebra $\mathfrak{l}_\ell \cap \mathfrak{b}$ of $\mathfrak{l}$ under the natural projection $\mathfrak{g}' \to \mathfrak{l}_\ell$. Note that V is a sum (not necessarily direct) of $(\mathfrak{b} \cap \mathfrak{l}_\ell)$-highest weight $\mathfrak{l}_\ell$-modules. Moreover, f_ℓ acts locally nilpotently on v_λ iff f_ℓ acts nilpotently on each of the (1- or ε-dimensional) highest weight spaces of these $\mathfrak{l}_\ell$-modules. Indeed, it is a tautology that if f_ℓ acts nilpotently on all those highest weight spaces, then f_ℓ acts locally nilpotently on v_λ. Conversely, let f_ℓ act locally nilpotently on v_λ. Then, by Lemma 1, f_ℓ acts locally finitely on V, i.e. necessarily locally nilpotently on the whole module V and, in particular, it acts nilpotently on the highest weight spaces of all summands in the decomposition of V into a sum of $\mathfrak{l}_\ell$-modules. Fix W to be one of these modules. W is a $(\mathfrak{b} \cap \mathfrak{l}_\ell)$-highest weight $\mathfrak{l}_\ell$-module with highest weight space v_ν (where $\dim v_\nu$ equals 1 or ε). Extend W to a $\mathfrak{b}'$-highest weight $\mathfrak{g}'$-module W', setting the action of $\mathfrak{i}$ on W' to be trivial. Since W could be chosen to be any of the $\mathfrak{l}_\ell$-modules in the decomposition of V, it is clear that in order to prove the Theorem it remains to establish only that f_ℓ acts nilpotently on v_ν iff $\alpha_\ell \in < \Sigma_{\mathfrak{b}'}^{W'} >_{\mathbb{R}}$.

To prove the latter statement we will show first that if f_ℓ acts nilpotently on v_ν then $\alpha_\ell \in < \Sigma_{\mathfrak{b}'}^{W'} >_{\mathbb{R}}$, i.e. that if f_ℓ acts nilpotently on v_ν, then so do f_{ℓ^i} for all i with $c^i \neq 0$. Fix i^0 with $c^{i^0} \neq 0$. We will establish by induction on $\mathrm{ht}\alpha_\ell := \sum_{i=1}^n c^i$ that $f_{\ell^{i^0}}$ acts nilpotently on v_ν. Let $f_\ell^N \cdot v_\nu = 0$. If $\mathrm{ht}\alpha_\ell = 1$, $\alpha_\ell \in \Sigma_{\mathfrak{b}}$ and thus $\alpha_\ell = \beta^{i^0}$, so there is nothing to prove. Suppose that if $\mathrm{ht}\alpha_\ell < r$ and f_ℓ acts nilpotently on v_ν then $f_{\ell^{i^0}}$ acts nilpotently on v_ν. Let $\mathrm{ht}\alpha_\ell = r$. There are two possible cases:

1. $[f_\ell, e_{\ell^{i^0}}] = 0$. Then there exists $i' \neq i^0$ such that $[f_\ell, e_{\ell^{i'}}] \neq 0$. Let k be determined by the condition $(\mathrm{ad}\, e_{\ell^{i'}})^k \cdot f_\ell \neq 0$ but $(\mathrm{ad}\, e_{\ell^{i'}})^{k+1} \cdot f_\ell = 0$. We claim that $\mathbb{R}(\alpha_\ell - k\beta^{i'})$ is a finite line of $\mathfrak{g}'$. Indeed, since $\mathfrak{g}'$ is a direct sum of finite-dimensional $(\mathfrak{g}')^{\ell^{i'}}$-modules, the assumption that $\mathbb{R}(\alpha_\ell - k\beta^{i'})$ is an infinite line of $\mathfrak{g}'$ would imply that ℓ is an infinite line of $\mathfrak{g}'$ as well, which is contradiction. Therefore

$$e_{\ell^{i^0}}^{kN} \cdot f_\ell^N = (N!)^k c^N f^N_{\mathbb{R}(\alpha_\ell - k\beta^{i'})} (\mathrm{mod}\ U(\mathfrak{g}) \cdot \mathfrak{n}^+) \tag{6}$$

and hence $f^N_{\mathbb{R}(\alpha_\ell - s\beta^{i'})} \cdot v_\nu = 0$. Since $\mathbb{R}(\alpha_\ell - k\beta^{i'})$ is a finite line and $\mathrm{ht}(\alpha_\ell - k\beta^{i'}) < r$, an obvious induction argument shows that $f_{\ell^{i^0}}$ acts nilpotently on v_ν.

2. $[f_\ell, e_{\ell^{i^0}}] \neq 0$. Consider the β^{i^0}-string through α_ℓ, and let $\alpha_\ell + p\beta^{i^0}$ and $\alpha_\ell - q\beta^{i^0}$ be its end points. As in the previous case we prove that both $\mathbb{R}(\alpha_\ell + p\beta^{i^0})$ and $\mathbb{R}(\alpha_\ell - q\beta^{i^0})$ are finite lines of $\mathfrak{g}'$ and that $V_{\mathfrak{b}'}(v_\nu)$ is $\mathbb{R}(\alpha_\ell + p\beta^{i^0})$-integrable and $\mathbb{R}(\alpha_\ell - q\beta^{i^0})$-integrable. Moreover, it is easy to check that the subalgebra of $\mathfrak{g}'$ generated by $e_{\mathbb{R}(\alpha_\ell - q\beta^{i^0})}, f_{\mathbb{R}(\alpha_\ell - q\beta^{i^0})}, e_{\ell^{i^0}}$ and $f_{\ell^{i^0}}$ is a subalgebra of a rank two Kac-Moody algebra. Set $x := f_{\mathbb{R}(\alpha_\ell + p\beta^{i^0})}, y := e_{\mathbb{R}(\alpha_\ell + p\beta^{i^0})}, z := [e_{\ell^{i^0}}, x]$ and $t := [f_{\ell^{i^0}}, y]$. Then the vectors $[x, y]$ and $[z, t]$ are necessarily linearly independent. Furthermore, $[t, x]$ is proportional to $f_{\ell^{i^0}}$, and we claim that $[t, x] \neq 0$. Indeed, the assumption that $[t, x] = 0$ would lead to

$$0 = [e_{\ell^{i^0}}, [t, x]] = [[e_{\ell^{i^0}}, t], x] - [[e_{\ell^{i^0}}, x], t] = (\alpha_\ell + p\beta^{i^0})(h_{\ell^{i^0}})[y, x] - [z, t],$$

which contradicts the linear independence of $[y, x]$ and $[z, t]$. But since x acts nilpotently on v_ν, a formula similar to (6) implies also that $f_{\ell^{i^0}}$ acts nilpotently on v_ν. This completes the proof of the statement that $\alpha_\ell \in < \Sigma_{\mathfrak{b}'}^{W'} >_{\mathbb{R}}$ whenever f_ℓ acts nilpotently on v_ν.

To prove the converse, i.e. that f_ℓ acts nilpotently on v_ν whenever $\alpha_\ell \in < \Sigma_{\mathfrak{b}'}^{W'} >_{\mathbb{R}}$, note that since $\mathfrak{g}'$ is a Kac-Moody algebra there exists a chain $\mathfrak{b}' = (\mathfrak{b}')^1, \dots, (\mathfrak{b}')^k = \mathfrak{b}''$ of Borel subalgebras of $\mathfrak{g}'$, such that $\mathbb{R}\alpha_\ell$ is a simple line of $\mathfrak{b}''$ and W' is a $(\mathfrak{b}')^j$-highest weight $\mathfrak{g}'$-module for every $1 \leq j \leq k$. Moreover, α_ℓ appears with a non-zero coefficient in the decomposition of some of the simple roots of $\mathfrak{b}'$ as a sum of simple roots of $\mathfrak{b}''$. Therefore the already proved part of the Theorem implies also that f_ℓ acts nilpotently on v_ν if $\alpha_\ell \in < \Sigma_{\mathfrak{b}'}^{W'} >_{\mathbb{R}}$. □

Remark. In the case when $\mathfrak{g}$ is finite-dimensional and V is irreducible the claim of the Theorem is known to be a non-difficult corollary of Gabber's theorem, see the more general Corollary 2.7 in Fernando's paper [Fe]. In the same paper Fernando has found an analogue of this result for non-highest weight irreducible modules with finite-dimensional weight spaces, Proposition 4.17. In the recent paper [DMP] Fernando's result has been extended to arbitrary irreducible generalized weight modules of finite-dimensional Lie superalgebras, Corollary 5.2. Our Theorem can be viewed as an infinite-dimensional version of this result for highest weight modules.

3. Examples

We will complete the paper by discussing several types of weak Kac-Moody superalgebras and their respective Borel subsuperalgebras which admit a weak basis.

3.1. **Finite-dimensional superalgebras.** Every finite-dimensional Lie superalgebra is a weak Kac-Moody superalgebra. Moreover, every Borel subsuperalgebra of a finite-dimensional Lie superalgebra admits a weak basis. Indeed, it is immediate to verify that if $\mathfrak{b}$ is a Borel subsuperalgebra and $\Sigma_{\mathfrak{b}}$ is a root basis for the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}_0^{ss}$ of the semi-simple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$, then $\Sigma_{\mathfrak{b}}$ is a weak basis of $\mathfrak{b}$.

3.2. **Kac-Moody algebras.** Every Kac-Moody algebra is a weak Kac-Moody algebra. However not every Borel subalgebra of a Kac-Moody algebra admits a weak basis. We have

Proposition 1. *a) A Borel subalgebra $\mathfrak{b}$ of a Kac-Moody algebra $\mathfrak{g}$ admits a weak basis iff $\mathfrak{b}$ is standard and there is a regular oriented flag F in $\mathbb{R}\Delta$ (see* 1.3*) so that $F^1 \cap Z = \{0\}$, Z being the imaginary cone of $\mathfrak{g}$ (see* [K2] *for the definition of imaginary cone).*

b) If $\mathfrak{b}$ admits a weak basis, then a weak basis of $\mathfrak{b}$ is nothing but a root basis of $\mathfrak{b}$, see [K2].

Proof. An exercise on the root systems of Kac-Moody algebras using Propositions 5.8 and 5.9 from [K2]. □

Noting that for an affine Lie algebra the imaginary cone Z coincides with the infinite line, we see that a Borel subalgebra of an affine Lie algebra admits a basis iff it is standard. This observation shows that for affine Lie algebras our Theorem is nothing but Proposition 3 in [DP] which is announced there without proof.

Finally, the reader will check that for any Kac-Moody algebra the Theorem is equivalent to the following statement: For every $\mathfrak{b}$-highest weight $\mathfrak{g}$-module V ($\mathfrak{b}$ being a Borel subalgebra of $\mathfrak{g}$ which admits a weak basis), there is a unique parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, containing $\mathfrak{b}$, such that the $sl(2)$-lines ℓ for which V is ℓ-integrable are precisely the $sl(2)$-lines of $\mathfrak{p}$.

3.3. **Affine superalgebras.** All affine Lie superalgebras considered in [DP] are weak Kac-Moody superalgebras. A Borel subsuperalgebra of any of these Lie superalgebras admits a weak basis iff it is standard. (However, obviously the weak bases introduced in the present paper do not coincide with the bases considered in [DP].) The main Theorem in section 2 leads to a quicker and more natural proof of Theorem 2 from [DP].

3.4. **Direct limit Lie algebras.** The direct limit Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$ are weak Kac-Moody algebras. Below we recall the definitions of those Lie algebras, characterize their Borel subalgebras, and give a criterion for the existence of a weak basis.

Consider the Lie algebra $gl(\infty)$ of infinite matrices $A = (a^{ij})_{i,j \in \mathbb{Z}}$ with finitely many non-zero entries, the Lie bracket being $[A, B] = AB - BA$. We fix the Cartan subalgebra $< E^{ii} >_{\mathbb{C}}$ of $gl(\infty)$, where E^{ij} denotes the matrix whose unique non-zero matrix element is $e^{ij} = 1$, and define $\varepsilon^k \in (< E^{ii} >_{\mathbb{C}})^*$, $k \in \mathbb{Z}$, by setting $\varepsilon^k(E^{ii}) := \delta^{ki}$. We then consider the following subalgebras of $gl(\infty)$:

$A(\infty) = sl(\infty) := \{A \in gl(\infty) \,|\, \mathrm{tr} A = 0\}$,

$B(\infty) := \{A \in gl(\infty) \,|\, RA = -A^t R \text{ for } R = (r^{ij}),\ r^{ij} = \delta^{i,-j}\}$,

$C(\infty) := \{A \in gl(\infty) \,|\, SA = -A^t S \text{ for } S = (s^{ij}),\ s^{ij} = -\mathrm{sgn}(i)\delta^{i,-j-1}\}$,

where $\mathrm{sgn}(i) := \begin{cases} -1 & \text{for} \quad i < 0 \\ 1 & \text{for} \quad i \geq 0 \end{cases}$,

$D(\infty) := \{A \in gl(\infty) \,|\, TA = -A^t T, \text{ where } T = (t^{ij}),\ t^{ij} = \delta^{i,-j-1}\}$.

Let $h^i := E^{ii} - E^{-i,-i}$ for $i \geq 1$ and let $k^i := E^{ii} - E^{-(i+1),-(i+1)}$ for $i \geq 0$. Define $\beta^j \in (< h^i >_{\mathbb{C}})^*$ and $\gamma^j \in (< k^i >_{\mathbb{C}})^*$ by setting $\beta^j(h^i) := \delta^{ij}, \gamma^j(k^i) := \delta^{ij}$. The Cartan subalgebras of the Lie algebras under consideration are:

$A(\infty) \quad : \quad \{< E^{ii} - E^{i-1,i-1} >_{\mathbb{C}} \,|\, i \in \mathbb{Z}\}$;

$B(\infty) \quad : \quad \{< h^i >_{\mathbb{C}} \,|\, i \geq 1\}$;

$C(\infty)$ and $D(\infty) \quad : \quad \{< k^i >_{\mathbb{C}} \,|\, i \geq 0\}$.

The corresponding sets of roots are:

$A(\infty) \quad : \quad \{\varepsilon^i - \varepsilon^j \,|\, i \neq j\}$,

$B(\infty) \quad : \quad \{\pm\beta^i, \quad \pm\beta^i \pm \beta^j \,|\, 1 \leq i \neq j\}$,

$C(\infty) \quad : \quad \{\pm 2\gamma^i, \quad \pm\gamma^i \pm \gamma^j \,|\, 0 \leq i \neq j\}$,

$D(\infty) \quad : \quad \{\pm\gamma^i \pm \gamma^j \,|\, 0 \leq i \neq j\}$.

Throughout section 3.4 $\mathfrak{g}$ will denote one of the Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ or $D(\infty)$, and $\mathfrak{h}$ will be the fixed Cartan subalgebra of $\mathfrak{g}$. It turns out that every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ (with $\mathfrak{b} \supset \mathfrak{h}$) is standard. More precisely, we have

Proposition 2. *Let $\mathfrak{b} \supset \mathfrak{h}$ be a Borel subalgebra of $\mathfrak{g}$. There exists (a not necessarily unique) $\varphi \in (\mathbb{R}\Delta)^*$, such that for any $\alpha \in \Delta$ one has $\alpha \in \Delta^+$ iff $\varphi(\alpha) > 0$.*

Proof. Consider first the cases $\mathfrak{g} = B(\infty)$, $C(\infty)$, $D(\infty)$ and define $\mathrm{sgn}\beta^i$ and $\mathrm{sgn}\gamma^i$ as explained below. For $\mathfrak{g} = B(\infty)$, set $\mathrm{sgn}\beta^i := \pm 1$ iff $\beta^i \in \Delta^{\pm}$. For $\mathfrak{g} = C(\infty)$, set $\mathrm{sgn}\gamma^i := \pm 1$ iff $2\gamma^i \in \Delta^{\pm}$. For $\mathfrak{g} = D(\infty)$, consider any pair of non-negative integers (j, k). Exactly two of the roots $\pm\gamma^j \pm \gamma^k$ belong to Δ^+. If both $\gamma^j + \gamma^k$ and $\gamma^j - \gamma^k$ belong to $\Delta^{\pm}$, put $\mathrm{sgn}\gamma^j := \pm 1$. If both $\gamma^j + \gamma^k$ and $-\gamma^j + \gamma^k$ belong to $\Delta^{\pm}$, put $\mathrm{sgn}\gamma^k := \pm 1$. An immediate verification shows that this definition of $\mathrm{sgn}\gamma^i$ is never contradictory but may be incomplete leaving $\mathrm{sgn}\gamma^{i_0}$ undefined for at most one i_0. In this latter case we choose the missing sign arbitrarily.

We put next $|\beta^i| := (\mathrm{sgn}\beta^i)\beta^i$ and $|\gamma^i| := (\mathrm{sgn}\gamma^i)\gamma^i$.

Considering now simultaneously the four cases $\mathfrak{g} = A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$, we set respectively

$\tilde{\varphi}(\varepsilon^i) := \sum_{\{j \in \mathbb{Z} | \ \varepsilon^i - \varepsilon^j \in \Delta^+\}} \frac{1}{2^{|j|}}$;

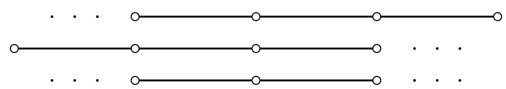
$\tilde{\varphi}(\beta^i) := \mathrm{sgn}\beta^i(1 + \sum_{\{j \in \mathbb{Z}_+ | \ |\beta^i| - |\beta^j| \in \Delta^+\}} \frac{1}{2^j})$;

$\tilde{\varphi}(\gamma^i) := \mathrm{sgn}\gamma^i(1 + \sum_{\{j \in \mathbb{Z}_+ | \ |\gamma^i| - |\gamma^j| \in \Delta^+\}} \frac{1}{2^j})$

and extend $\tilde{\varphi}$ by linearity.

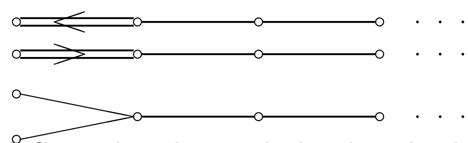
The reader will finish the proof by verifying that $\varphi = \tilde{\varphi}_{|\mathbb{R}\Delta}$ is always a linear function as desired. □

It remains to describe all Borel subalgebras of $\mathfrak{g}$ which admit a weak basis. Let $\mathfrak{g} = A(\infty)$. Let φ be the function constructed in Proposition 2. It is easy to check that $\mathbb{R}(\varepsilon^i - \varepsilon^j)$ is a simple line for $\mathfrak{b}$ iff there is no k so that the real number $\varphi(\varepsilon^k)$ is between the numbers $\varphi(\varepsilon^i)$ and $\varphi(\varepsilon^j)$. This implies that $\mathfrak{b}$ admits a weak basis iff the sequence $\{\varphi(\varepsilon^i)\}$ has no limit points other than $\inf \varphi(\varepsilon^i)$ and $\sup \varphi(\varepsilon^i)$. (Note that $|\varphi(\varepsilon^i)| \leq 2$ and thus $\inf \varphi(\varepsilon^i)$ and $\sup \varphi(\varepsilon^i)$ clearly exist.) There are three possible cases: $\inf \varphi(\varepsilon^i)$ is the only limit point of $\{\varphi(\varepsilon^i)\}$, $\sup \varphi(\varepsilon^i)$ is the only limit point of $\{\varphi(\varepsilon^i)\}$ and both $\inf \varphi(\varepsilon^i)$ and $\sup \varphi(\varepsilon^i)$ are limit points of $\{\varphi(\varepsilon^i)\}$. The corresponding Dynkin diagrams are:



Let now $\mathfrak{g} = B(\infty), C(\infty)$ and $D(\infty)$. The only difference is that in these cases $\mathfrak{b}$ has a weak basis iff the only limit point of $\{|\varphi(\beta^i)|\}$ (respectively of $\{|\varphi(\gamma^i)|\}$) is $\sup|\varphi(\beta^i)|$ (resp. $\sup|\varphi(\gamma^i)|$), and hence all Borel subalgebras which admit a weak basis correspond to the following Dynkin diagrams:

$B(\infty)$:
$C(\infty)$:
$D(\infty)$:



In [BB] Yu. Bahturin and G. Benkart have calculated explicitly the highest weights of all irreducible integrable $\mathfrak{b}$-highest weight $\mathfrak{g}$-modules (where $\mathfrak{g} = A(\infty)$, $B(\infty)$, $C(\infty)$, $D(\infty)$) for some special Borel subalgebras $\mathfrak{b}$. A direct checking shows that these Borel subalgebras admit a weak basis. Bahturin and Benkart define Borel subalgebras in $\mathfrak{g}$ as a direct limit of Borel subalgebras of simple finite-dimensional subalgebras, but using Proposition 2 it is not difficult to verify that for $\mathfrak{g} = A(\infty), B(\infty), C(\infty), D(\infty)$ the class of Borel subalgebras considered in [BB] is the same as the class considered in the present paper.

For $\mathfrak{g} = A(\infty), B(\infty), C(\infty)$ and $D(\infty)$, the Theorem from section 2 is equivalent to the following statement: For every $\mathfrak{b}$-highest weight $\mathfrak{g}$-module V ($\mathfrak{b}$ being a Borel subalgebra of $\mathfrak{g}$ which admits a weak basis), there is a unique parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, containing $\mathfrak{b}$, such that the lines of $\mathfrak{p}$ are precisely those lines ℓ of $\mathfrak{g}$ for which V is ℓ-integrable.

3.5. **Direct limit Lie superalgebras.** Consider the Lie superalgebra $gl(\infty+\infty\varepsilon)$. It can be defined as follows. Declare (the elementary matrix) E^{ij} to be even iff $i \geq 0, j \geq 0$ or $i < 0, j < 0$, and E^{ij} to be odd iff $i \geq 0, j < 0$ or $i < 0, j \geq 0$. This gives a $\mathbb{Z}_2$-grading on the span $< E^{ij} >_{\mathbb{C}}$ (which by definition is the underlying vector space of $gl(\infty+\infty\varepsilon)$), and we define the Lie superbracket of $gl(\infty+\infty\varepsilon)$ as the supercommutator of matrices corresponding to this $\mathbb{Z}_2$-grading. There are several natural subsuperalgebras of $gl(\infty + \infty\varepsilon)$: $A(m + \infty\varepsilon)$, $A(\infty + \infty\varepsilon)$, $B(m + \infty\varepsilon)$, $B(\infty + \infty\varepsilon)$, $C(\infty)$, $D(m + \infty\varepsilon)$, $D(\infty + \infty\varepsilon)$, $\mathbf{p}(\infty)$ and $q(\infty)$. All of these Lie superalgebras are weak Kac-Moody superalgebras and moreover all their Borel subsuperalgebras (containing the intersection of the considered Lie superalgebra with the subsuperalgebra of diagonal matrices $< E^{ii} >_{\mathbb{C}}$) are standard. Since a Borel subsuperalgebra $\mathfrak{b}$ admits a weak basis iff $\mathfrak{b}_0$ admits a weak basis, the results of 3.4 provide also an explicit criterion for $\mathfrak{b}$ to admit a weak basis.

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