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Weight Modules of Direct Limit Lie Algebras

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0 Introduction

The purpose of this paper is to initiate a systematic study of the irreducible weight representations of direct limits of reductive Lie algebras and, in particular, of the classical simple direct limit Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$. We study arbitrary, not necessarily highest weight, irreducible weight modules and describe the supports of all such modules. The representation theory of the classical direct limit groups has been initiated in the pioneering works of G. Olshanskii [O1], [O2] and is now in an active phase (see the recent works of A. Habib [Ha], L. Natarajan [Na], K.-H. Neeb [Ne], and L. Natarajan, E. Rodriguez-Carrington, and J. A. Wolf [NRW]). Nevertheless, the structure theory of weight representations of the simple direct limit Lie algebras has until recently been still in its infancy as only highest weight modules have been discussed in written works (see the works of Yu. A. Bahturin and G. Benkart [BB], K.-H. Neeb [Ne], and T. D. Palev [P]).

Our approach is based mainly on the recent paper [DMP] in which a general method for studying the support of weight representations of finite-dimensional Lie al-

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gebras (and Lie superalgebras) was developed. We prove first that the shadow of any irreducible weight module M (over any root reductive direct limit Lie algebra, in particular over $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$) is well defined. This means that for a given root α, the intersection of the ray $\lambda + \mathbb{R}_+\alpha$ with the support of M, $\operatorname{supp} M$, is either finite for all $\lambda \in \operatorname{supp} M$ or infinite for all $\lambda \in \operatorname{supp} M$. Using this remarkable property of the support, we assign to M a canonical parabolic subalgebra $\mathfrak{p}_M$ of $\mathfrak{g}$ and then compare M with the irreducible quotient of a certain $\mathfrak{g}$-module induced from $\mathfrak{p}_M$. In the case of a finite-dimensional Lie algebra, the Fernando-Futorny parabolic induction theorem (see [Fe], [Fu], and [DMP]) states that M is always such a quotient and, moreover, that $\operatorname{supp} M$ is nothing but the support of the induced module. In the direct limit case, we show that this is no longer true but nevertheless, using the fact that the shadow is well defined, we present a description of the support of any irreducible weight module.

We discuss in more detail the following special cases: where $\operatorname{supp} M$ is finite in all root directions (these are the finite integrable irreducible modules and they are analogues of finite-dimensional irreducible modules over finite-dimensional Lie algebras); and the more general case where $\operatorname{supp} M$ is finite in at least one of each of two mutually opposite root directions. An interesting feature in the first case is that M is not necessarily a highest weight module; i.e., the analogues of finite-dimensional irreducible modules are already outside the class of highest weight modules. We present an explicit parametrization of all finite integrable irreducible modules. The modules corresponding to the second case are by definition pseudo highest weight modules. One of their interesting features is that, in general, they are not obtained by parabolic induction as in the Fernando-Futorny theorem.

1 Generalities on triangular decompositions and weight modules

The ground field is $\mathbb{C}$. The signs $\in\!\!\!\!+$ and $\ni\!\!\!\!+$ denote the semidirect sum of Lie algebras (if $\mathfrak{g} = \mathfrak{g}' \in\!\!\!\!+\, \mathfrak{g}''$, then $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$ as well as a $\mathfrak{g}''$-module). The superscript * always stands for dual space. We set $\mathbb{R}_+ := \{r \in \mathbb{R} \mid r \geq 0\}$, $\mathbb{R}_- := -\mathbb{R}_+$, $\mathbb{Z}_\pm := \mathbb{Z} \cap \mathbb{R}_\pm$, and the linear span with coefficients in $\mathbb{C}$ (respectively, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{R}_\pm$) is denoted by $\langle\ \ \rangle_\mathbb{C}$ (resp., by $\langle\ \ \rangle_\mathbb{R}$, $\langle\ \ \rangle_\mathbb{Z}$, $\langle\ \ \rangle_{\mathbb{R}_\pm}$). If $\mathfrak{g}$ is any Lie algebra and M is a $\mathfrak{g}$-module, we call M *integrable* if and only if $\mathfrak{g}$ acts locally finitely on M; i.e., if for any $m \in M$ and any $g \in \mathfrak{g}$, the space $\langle m, g \cdot m, g^2 \cdot m, \ldots \rangle_\mathbb{C}$ is finite-dimensional. This terminology has been introduced by V. Kac; in [BB], integrable modules are called locally finite. If $\mathfrak{g}$ is a direct sum of Lie algebras, $\mathfrak{g} = \oplus_{s \in S} \mathfrak{g}^s$, and for each s, M^s is an irreducible $\mathfrak{g}^s$-module with a fixed nonzero vector $m^s \in M$, then $(\otimes_s M^s)(\otimes_s m^s)$ denotes the $\mathfrak{g}$-submodule of $\otimes_s M^s$ generated by $\otimes_s m^s$. It is easy to check that $(\otimes_s M^s)(\otimes_s m^s)$ is an irreducible $\mathfrak{g}$-module.

Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra* of $\mathfrak{g}$ is by definition a self-normalizing nilpotent Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We do not assume $\mathfrak{g}$ or $\mathfrak{h}$ to be finite-dimensional. A $\mathfrak{g}$-module M is a *generalized weight* $\mathfrak{g}$*-module* if and only if as an $\mathfrak{h}$-module, M decomposes as the direct sum $\oplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, where

$$M^\lambda := \{m \in M \mid h - \lambda(h) \text{ acts nilpotently on } m \text{ for every } h \in \mathfrak{h}\}.$$

We call M^λ the *generalized weight space of* M *of weight* λ. Obviously, a generalized weight $\mathfrak{g}$-module is integrable as an $\mathfrak{h}$-module. The *support* of M, $\operatorname{supp} M$, consists of all weights λ with $M^\lambda \neq 0$. A generalized weight module M is a *weight module* if and only if $\mathfrak{h}$ acts semisimply on M, i.e., if and only if each generalized weight space is isomorphic to the direct sum of one-dimensional $\mathfrak{h}$-modules.

Henceforth, $\mathfrak{g}$ will denote a Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ is a generalized weight module; i.e.,

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}^\alpha). \tag{1}$$

The generalized weight spaces $\mathfrak{g}^\alpha$ are by definition the *root spaces* of $\mathfrak{g}$; $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}^\alpha \neq 0\}$ is the set of *roots* of $\mathfrak{g}$; and the decomposition (1) is the *root decomposition* of $\mathfrak{g}$.

Below we recall the definitions of a triangular decomposition of Δ and of a Borel subalgebra for an arbitrary Lie algebra $\mathfrak{g}$ with a root decomposition. For more details, see [DP2]. A decomposition of Δ,

$$\Delta = \Delta^+ \sqcup \Delta^-, \tag{2}$$

is a *triangular decomposition* of Δ if and only if the cone $\langle \Delta^+ \cup -\Delta^- \rangle_{\mathbb{R}_+}$ [1] (or equivalently, its opposite cone $\langle -\Delta^+ \cup \Delta^- \rangle_{\mathbb{R}_+}$) contains no (real) vector subspace. Equivalently, (2) is a triangular decomposition if the following is a well-defined $\mathbb{R}$-linear partial order on $\langle \Delta \rangle_{\mathbb{R}}$:

$$\eta \geq \mu \Leftrightarrow \eta = \mu + \sum_{i=1}^{n} c_i \alpha_i \text{ for some } \alpha_i \in \Delta^+ \cup -\Delta^- \text{ and } c_i \in \mathbb{R}_+.$$

(A partial order is $\mathbb{R}$-*linear* if it is compatible with addition and multiplication by positive real numbers, and if multiplication by negative real numbers changes order direction. In what follows, unless explicitly stated that it is partial, an order will always be assumed linear, i.e., such that $\alpha \neq \beta$ implies $\alpha < \beta$ or $\alpha > \beta$. In particular, an $\mathbb{R}$-*linear order* is by definition an order which is in addition $\mathbb{R}$-linear.) Any regular real hyperplane H in

[1] We use the term cone as a synonym for an $\mathbb{R}_+$-invariant additive subset of a real vector space.

$\langle\Delta\rangle_{\mathbb{R}}$ (i.e., a codimension one linear subspace H with $H \cap \Delta = \emptyset$) determines exactly two triangular decompositions of Δ: we first assign (in an arbitrary way) the sign $+$ to one of the two connected components of $\langle\Delta\rangle_{\mathbb{R}}\backslash H$, and the sign $-$ to the other. $\Delta^{\pm}$ are then by definition the subsets of Δ which belong, respectively, to the "positive" and the "negative" connected components of $\langle\Delta\rangle_{\mathbb{R}}\backslash H$. In general, not every triangular decomposition of Δ arises in this way (see [DP1]). Nevertheless, it is true that every triangular decomposition is determined by an (not unique) oriented maximal chain of vector subspaces in $\langle\Delta\rangle_{\mathbb{R}}$: see the Appendix, where we establish the precise interrelationship between oriented maximal chains in $\langle\Delta\rangle_{\mathbb{R}}$, $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$, and triangular decompositions of Δ.

A Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is by definition a *Borel subalgebra* of $\mathfrak{g}$ if and only if $\mathfrak{b} = \mathfrak{h} \supsetplus (\oplus_{\alpha\in\Delta^+}\mathfrak{g}^{\alpha})$ for some triangular decomposition $\Delta = \Delta^+ \sqcup \Delta^-$. In what follows, a Borel subalgebra of $\mathfrak{g}$ always means a Borel subalgebra containing the fixed Cartan subalgebra $\mathfrak{h}$. Adopting terminology from affine Kac-Moody algebras, we call a Borel subalgebra *standard* if and only if it corresponds to a triangular decomposition that can be determined by a regular hyperplane H in $\langle\Delta\rangle_{\mathbb{R}}$.

If $\mathfrak{b}$ is a Borel subalgebra and $\lambda \in \mathfrak{h}^*$, then the *Verma module* $\tilde{V}_{\mathfrak{b}}(\lambda)$ *with* $\mathfrak{b}$-*highest weight* λ is by definition the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, $\mathbb{C}_\lambda$ being the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts via λ. Any quotient of $\tilde{V}_{\mathfrak{b}}(\lambda)$ is by definition a $\mathfrak{b}$-*highest weight module*. Furthermore, $\tilde{V}_{\mathfrak{b}}(\lambda)$ has a unique maximal proper $\mathfrak{g}$-submodule $I_{\mathfrak{b}}(\lambda)$, and $V_{\mathfrak{b}}(\lambda) := \tilde{V}_{\mathfrak{b}}(\lambda)/I_{\mathfrak{b}}(\lambda)$ is by definition the *irreducible* $\mathfrak{b}$-*highest weight* $\mathfrak{g}$-*module with highest weight* λ.

2 Direct limits of reductive Lie algebras

A homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras with root decomposition is a *root homomorphism* if and only if $\varphi(\mathfrak{h}) \subset \mathfrak{h}'$ ($\mathfrak{h}$ and $\mathfrak{h}'$ being the corresponding fixed Cartan subalgebras) and φ maps any root space of $\mathfrak{g}$ into a root space of $\mathfrak{g}'$. Let

$$\mathfrak{g}_1 \xrightarrow{\varphi_1} \mathfrak{g}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} \mathfrak{g}_n \xrightarrow{\varphi_n} \dots \tag{3}$$

be a chain of homomorphisms of finite-dimensional Lie algebras, and let $\mathfrak{g} := \varinjlim \mathfrak{g}_n$ be the direct limit Lie algebra. We say that $\mathfrak{g}$ is a *root direct limit of the system* (3) if and only if φ_n is a root homomorphism for every n. In the latter case, $\mathfrak{h}_n$ denotes the fixed Cartan subalgebra of $\mathfrak{g}_n$. We define a Lie algebra $\mathfrak{g}$ to be a *root direct limit Lie algebra* if and only if $\mathfrak{g}$ is a root direct limit of some direct system of the form (3). Furthermore, a nonzero Lie algebra $\mathfrak{g}$ is a *root reductive direct limit of the system* (3) if and only if all $\mathfrak{g}_n$ are reductive; and $\mathfrak{g}$ is a *root simple direct limit Lie algebra* if and only if $\mathfrak{g}$ is a root direct limit of a system (3) in which all $\mathfrak{g}_n$ are simple.

Proposition 1. Let $\mathfrak{g}$ be a root direct limit Lie algebra. Then $\mathfrak{h} := \varinjlim \mathfrak{h}_n$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ has a root decomposition with respect to $\mathfrak{h}$ such that $\Delta = \varinjlim \Delta_n$, where Δ_n and Δ are, respectively, the roots of $\mathfrak{g}_n$ and $\mathfrak{g}$. If all root homomorphisms φ_n are embeddings, one has simply $\mathfrak{h} = \cup_n \mathfrak{h}_n$ and $\Delta = \cup_n \Delta_n$. □

Proof. The proof is a trivial exercise. ■

Every simple finite-dimensional Lie algebra $\mathfrak{g}$ is a root simple direct limit Lie algebra: we set $\mathfrak{g}_n := \mathfrak{g}$ and $\varphi_n := \mathrm{id}_{\mathfrak{g}}$. To define the simple Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$, it suffices to let $\mathfrak{g}_n$ be the corresponding rank n simple Lie algebra and to request that all φ_n be injective root homomorphisms, i.e., root embeddings. Indeed, one can check that in these cases, the direct limit Lie algebra does not depend up to isomorphism on the choice of root embeddings φ_n. More generally, Theorem 4.4 in [BB] implies that every infinite-dimensional root simple direct limit Lie algebra is isomorphic to one of the Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$, or $D(\infty)$. Note however, that for general root reductive direct limit Lie algebras, the direct limit Lie algebra can depend on the choice of root homomorphisms φ_n even if they are embeddings. Indeed, set, for instance, $\mathfrak{g}_n := A(2^n) \oplus B(2^n)$, and let $\varphi_n, \varphi'_n : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$ be root embeddings such that $\varphi_n(A(2^n)) \subset A(2^{n+1})$ and $\varphi_n(B(2^n)) \subset B(2^{n+1})$ but $\varphi'_n(A(2^n) \oplus B(2^n)) \subset B(2^{n+1})$. Then $\mathfrak{g} \simeq A(\infty) \oplus B(\infty)$ but $\mathfrak{g}' \simeq B(\infty)$.

Let $\mathfrak{g}$ be a root reductive direct limit Lie algebra and let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k}$. Then $\Delta_{\mathfrak{k}} \subset \Delta_{\mathfrak{g}}$ and we can set $\Delta_{\mathfrak{k}}^{ss} := \Delta_{\mathfrak{k}} \cap (-\Delta_{\mathfrak{k}})$. The Lie subalgebra $\mathfrak{k}^{ss} := [\oplus_{\alpha \in \Delta_{\mathfrak{k}}^{ss}} \mathfrak{k}^{\alpha}, \oplus_{\alpha \in \Delta_{\mathfrak{k}}^{ss}} \mathfrak{k}^{\alpha}]$ of $\mathfrak{k}$ is an analogue of the semisimple part of $\mathfrak{k}$ in the case where $\mathfrak{k}$ is not finite-dimensional.

Theorem 1. Let $\mathfrak{g}$ be a root reductive direct limit Lie algebra. Then we have the following.

(i) $\mathfrak{g} \simeq \mathfrak{g}^{ss} \Subset A$, where A is an abelian Lie algebra of finite or countable dimension. Furthermore, $\mathfrak{h} = \mathfrak{h}^{ss} \oplus A$, where $\mathfrak{h}^{ss} := \mathfrak{h} \cap \mathfrak{g}^{ss}$.

(ii) $\mathfrak{g}^{ss} \simeq \oplus_{s \in S} \mathfrak{g}^s$, where S is a finite or countable family of root simple direct limit Lie algebras $\mathfrak{g}^s$. □

Proof. (i) Decompose $\mathfrak{h}$ as $\mathfrak{h} = \mathfrak{h}^{ss} \oplus A$ for some vector space A. Then A is an abelian subalgebra of $\mathfrak{h}$ (since $\mathfrak{h}$ itself is abelian) of at most countable dimension, and using Proposition 1, one checks that $\mathfrak{g} \simeq \mathfrak{g}^{ss} \Subset A$.

(ii) Let $\mathfrak{g} = \varinjlim \mathfrak{g}_n$. Proposition 1 implies immediately that $\mathfrak{g}^{ss} = \varinjlim \mathfrak{g}_n^{ss}$. Furthermore, since a Cartan subalgebra $\mathfrak{h}_n \subset \mathfrak{g}_n$ is fixed for every n, there is a canonical decomposition

$$\mathfrak{g}_n = (\oplus_{t \in S_n} \mathfrak{g}^t) \oplus Z_n$$

such that all $\mathfrak{g}^t$ are simple, Z_n is abelian, and, for every $\alpha \in \Delta_n$, $\mathfrak{g}_n^{\alpha} \subset \mathfrak{g}^t$ for some $t \in S_n$. Then, for any $t \in S_n$, either $\varphi_n(\mathfrak{g}^t) = 0$ or there is $t' \in S_{n+1}$ so that $\varphi_n(\mathfrak{g}^t)$ is a nontrivial subalgebra of $\mathfrak{g}^{t'}$ (we assume that the sets S_n are pairwise disjoint). Put $S' := \cup_n\{t \in S_n \mid \varphi_{n,n'}(\mathfrak{g}^t) \neq 0 \text{ for every } n' > n\}$, where $\varphi_{n_1,n_2} := \varphi_{n_2} \circ \ldots \circ \varphi_{n_1}$ for $n_1 < n_2$. Introduce an equivalence relation $\sim$ on S' by setting $t_1 \sim t_2$ for $t_1 \in S_{n_1}$, $t_2 \in S_{n_2}$, if and only if there exists n such that $\varphi_{n_1,n}(\mathfrak{g}^{t_1})$ and $\varphi_{n_2,n}(\mathfrak{g}^{t_2})$ belong to the same simple component of $\mathfrak{g}_n$. Define S as the set of classes of $\sim$-equivalence. For every $s \in S$, the set $\{\mathfrak{g}^t\}_{t\in s}$ is partially ordered by the maps $\varphi_{n_1,n_2} : \mathfrak{g}^{t_1} \to \mathfrak{g}^{t_2}$. Let $\mathfrak{g}^s$ be the direct limit Lie algebra of a maximal chain of Lie algebras among $\{\mathfrak{g}^t\}_{t\in s}$ with respect to this partial order. Obviously, $\mathfrak{g}^s$ is a root simple direct limit Lie algebra, and it does not depend on the choice of the maximal chain. Finally, one checks easily that $\mathfrak{g}^{ss} = \varinjlim \mathfrak{g}_n^{ss} \simeq \oplus_{s\in S}\mathfrak{g}^s$. ■

Example 1. $\mathfrak{gl}(\infty)$ can be defined as the Lie algebra of all infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}_+}$ with finitely many nonzero entries. Then $\mathfrak{gl}(\infty) \simeq A(\infty) \in \mathbb{C}$, but $\mathfrak{gl}(\infty) \not\simeq A(\infty) \oplus \mathbb{C}$ as the center of $\mathfrak{gl}(\infty)$ is trivial.

Theorem 1 gives an almost explicit description of all root reductive direct limit Lie algebras. In particular, it implies that any root space of a root reductive direct limit Lie algebra has dimension one. Moreover, if $\pi : \mathfrak{h}^* \to (\mathfrak{h}^{ss})^*$ denotes the natural projection, then Theorem 1 implies that π induces a bijection between Δ and the set of roots of $\mathfrak{g}^{ss}$. Note also that for any $\lambda \in \mathfrak{h}^*$ and any $\alpha \in \Delta$, the scalar product $(\lambda|_{\mathfrak{h}_n}, \alpha|_{\mathfrak{h}_n})_n$ ($(\cdot,\cdot)_n$ being the standard bilinear form $\mathfrak{h}_n^* \times \mathfrak{h}_n^* \to \mathbb{C}$) has a fixed value for all n such that $\alpha|_{\mathfrak{h}_n} \neq 0$, and therefore we can set $(\lambda, \alpha) := (\lambda|_{\mathfrak{h}_{n'}}\ \alpha|_{\mathfrak{h}_n})$ for sufficiently large n. We put then

$$\langle \lambda, \alpha \rangle := \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

By definition, λ is an *integral weight of* $\mathfrak{g}$ if and only if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for every $\alpha \in \mathbb{Z}$; and λ is $\mathfrak{b}$-dominant, for a Borel subalgebra $\mathfrak{b} = \mathfrak{h}(\oplus_{\alpha\in\Delta} + \mathfrak{g}^\alpha)$, if and only if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$. The *Weyl group W of* $\mathfrak{g}$ is the group generated by all reflections σ_α: $\mathfrak{h}^* \to \mathfrak{h}^*$, $\alpha \in \Delta$, where $\sigma_\alpha := \lambda - \langle \lambda, \alpha \rangle \alpha$. By its very definition, W acts on $\mathfrak{h}^*$.

The exact relationship between irreducible generalized weight $\mathfrak{g}$-modules and irreducible generalized weight $\mathfrak{g}^{ss}$-modules (all of which automatically turn out to be weight modules) is established in the following proposition.

Proposition 2. (i) Every irreducible generalized weight $\mathfrak{g}$-module M is a weight module.

(ii) Every irreducible weight $\mathfrak{g}$-module M is irreducible as a (weight) $\mathfrak{g}^{ss}$-module.

(iii) Given any irreducible weight $\mathfrak{g}^{ss}$-module M^{ss}, every $\lambda \in \mathfrak{h}^*$ with $\pi(\lambda) \in \operatorname{supp} M^{ss}$ defines a unique structure of an irreducible weight $\mathfrak{g}$-module on M^{ss} which extends the $\mathfrak{g}^{ss}$-module structure on M^{ss}. If $M^{ss}(\lambda)$ denotes the resulting $\mathfrak{g}$-module, then $M^{ss}(\lambda) \simeq M^{ss}(\lambda')$ if and only if $\pi(\lambda - \lambda') = \sum_i c_i\pi(\alpha_i)$ for some $\alpha_i \in \Delta$ implies $\lambda - \lambda' = \sum_i c_i\alpha_i$. □

Proof. (i) Let U^0 denote the subalgebra of the enveloping algebra $U(\mathfrak{g})$ generated by all monomials of weight zero. Since $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$, the symmetric algebra $S(\mathfrak{h})$

belongs to the center of U^0. Furthermore, any generalized weight space M^λ is an irreducible U^0-module and, by a general version of Schur's Lemma, $S(\mathfrak{h})$ acts via a scalar on M^λ. Therefore, M is a semisimple $\mathfrak{h}$-module; i.e., M is a weight module.

(ii) Follows immediately from (i) and Theorem 1(i).

(iii) Let any $a \in A$ act on the weight space M^μ of M^{ss} via multiplication by $\lambda(a) + \sum_i \alpha_i(a)$, where $\mu - \pi(\lambda) = \sum_i c_i \pi(\alpha_i)$. It is straightforward to verify that this equips M^{ss} with a well-defined $\mathfrak{g}$-module structure.

The isomorphism criterion is also an easy exercise. ■

Our main objective in this paper is the study of the irreducible weight modules over root reductive direct limit Lie algebras. The case of finite-dimensional reductive Lie algebras is discussed in particular in [Fe], [Fu], [DMP], [M].

In the rest of this section, we study the structure of the root simple direct limit Lie algebras, and $\mathfrak{g}$ stands for $A(\infty)$, $B(\infty)$, $C(\infty)$, or $D(\infty)$. If ε_i are the usual linear functions on the Cartan subalgebras of the simple finite-dimensional Lie algebras (see for example, [B] or [Hu]), one can let i run from 1 to ∞ and then (using Proposition 1) verify the following list of roots:

$$A(\infty): \quad \Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\},$$

$$B(\infty): \quad \Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\},$$

$$C(\infty): \quad \Delta = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\},$$

$$D(\infty): \quad \Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}.$$

Every sequence of complex numbers $\{\lambda^n\}_{n=1,2,\ldots}$ determines a weight λ of $\mathfrak{g}$ by setting $\lambda(\varepsilon_n) := \lambda^n$. For $\mathfrak{g} = B(\infty)$, $C(\infty)$, or $D(\infty)$, every weight λ of $\mathfrak{g}$ recovers the sequence $\{\lambda^n := \lambda(\varepsilon_n)\}_{n=1,2,\ldots}$; for $\mathfrak{g} = A(\infty)$, the weight λ recovers the sequence $\{\lambda^n := \lambda(\varepsilon_n)\}_{n=1,2,\ldots}$ up to an additive constant only.

It is proved in [DP2, Proposition 2] that all Borel subalgebras of $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$ are standard. More precisely, for every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, there is a linear function $\varphi : \langle\Delta\rangle_{\mathbb{R}} \to \mathbb{R}$ such that $\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\varphi(\alpha)>0}\mathfrak{g}^\alpha)$. Define an order (or a partial order) on the set $\{0\} \cup \{\pm\varepsilon_i\}$ to be $\mathbb{Z}_2$-*linear* if and only if multiplication by -1 reverses the order. Then [DP2, Proposition 2] is essentially equivalent to the following statement.

Proposition 3. If $\mathfrak{g} = A(\infty)$, then there is a bijection between Borel subalgebras of $\mathfrak{g}$ and orders on the set $\{\varepsilon_i\}$.

If $\mathfrak{g} = B(\infty)$ or $C(\infty)$, there is a bijection between Borel subalgebras of $\mathfrak{g}$ and $\mathbb{Z}_2$-linear orders on the set $\{0\} \cup \{\pm\varepsilon_i\}$.

If $\mathfrak{g} = D(\infty)$, there is a bijection between Borel subalgebras of $\mathfrak{g}$ and $\mathbb{Z}_2$-linear orders on the set $\{0\} \cup \{\pm\varepsilon_i\}$ with the property that, if there is a minimal positive element with respect to this order, then this element is of the form ε_i. □

Proof. The pullback via φ of the standard order on $\mathbb{R}$ determines an order on $\{0\} \cup \Delta$ which induces an order, respectively, on $\{\varepsilon_i\}$ or $\{0\} \cup \{\pm\varepsilon_i\}$, as desired. Conversely, for every order $\{\varepsilon_i\}$, or, respectively, for every $\mathbb{Z}_2$-linear order on $\{0\} \cup \{\pm\varepsilon_i\}$ as in the proposition, there exists a (nonunique) linear function $\varphi : \langle\{\varepsilon_i\}\rangle_{\mathbb{R}} \to \mathbb{R}$ such that $\Delta^+ = \varphi^{-1}(\mathbb{R}^+) \cap \Delta$. ■

The result of Proposition 3 can be found in an equivalent form in [Ne]. The case of $A(\infty)$ is due to V. Kac (unpublished).

Example 2. $A(\infty)$ is naturally identified with the Lie algebra of traceless infinite matrices $(a_{i,j})_{i,j\in\mathbb{Z}_+}$ with finitely many nonzero entries, as well as with the Lie algebra of traceless double infinite matrices $(a_{i,j})_{i,j\in\mathbb{Z}}$ with finitely many nonzero entries. The respective algebras of upper triangular matrices are nonisomorphic Borel subalgebras of $A(\infty)$ corresponding, respectively, to the orders $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$ and $\cdots > \varepsilon_6 > \varepsilon_4 > \varepsilon_2 > \varepsilon_1 > \varepsilon_3 > \varepsilon_5 > \cdots$.

A *parabolic subalgebra* of a root reductive direct limit Lie algebra is by definition a Lie subalgebra containing a Borel subalgebra. (The general definition of a parabolic subalgebra of a Lie algebra with root decomposition is given in the Appendix.) Here is an explicit description of all parabolic subalgebras of $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$.

Proposition 4. If $\mathfrak{g} = A(\infty)$, then there is a bijection between parabolic subalgebras of $\mathfrak{g}$ and partial orders on the set $\{\varepsilon_i\}$.

If $\mathfrak{g} = B(\infty)$ or $C(\infty)$, then there is a bijection between parabolic subalgebras of $\mathfrak{g}$ and $\mathbb{Z}_2$-linear partial orders on the set $\{0\} \cup \{\pm\varepsilon_i\}$.

If $\mathfrak{g} = D(\infty)$, then there is a bijection between parabolic subalgebras of $\mathfrak{g}$ and $\mathbb{Z}_2$-linear partial orders on the set $\{0\} \cup \{\pm\varepsilon_i\}$ with the property that if ε_i is not comparable with 0 for some i (i.e., neither $\varepsilon_i > 0$ nor $\varepsilon_i < 0$), then ε_j is also not comparable with 0 for some $j \neq i$. □

Proof. Given a partial order on $\{\varepsilon_i\}$ (respectively, a $\mathbb{Z}_2$-linear partial order on $\{0\} \cup \{\pm\varepsilon_i\}$ with the additional property for $\mathfrak{g} = D(\infty)$), it determines a unique partial order $>$ on the set $\{0\} \cup \Delta$. Put $\mathfrak{p}_> := \mathfrak{h} \oplus (\oplus_{\alpha>0 \text{ or } \alpha \text{ not comparable with } 0}\mathfrak{g}^\alpha)$. Then $\mathfrak{p}_>$ is the parabolic subalgebra corresponding to the initial partial order. Conversely, let $\mathfrak{p}$ be a parabolic subalgebra. For $\alpha \in \Delta$, set $\alpha >_{\mathfrak{p}} 0$ if and only if $\mathfrak{g}^\alpha \subset \mathfrak{p}$ but $\mathfrak{g}^{-\alpha} \not\subset \mathfrak{p}$. Using the explicit form of Δ, it is easy to verify that this determines a unique partial order on $\{\varepsilon_i\}$ (respectively, on $\{0\} \cup \{\pm\varepsilon_i\}$, as desired). ■

The next proposition describes $\mathfrak{p}^{ss}$ for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g} = A(\infty)$, $B(\infty)$, $C(\infty)$, or $D(\infty)$; it will be used in Section 3.

Proposition 5. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$.

(i) $\mathfrak{p}^{ss}$ is isomorphic to a direct sum of simple Lie algebras, each of which is one of the following:

- $A(n)$ or $A(\infty)$, if $\mathfrak{g} = A(\infty)$;
- $A(n)$, $A(\infty)$, $B(n)$, or $B(\infty)$ with at most one simple component isomorphic to $B(n)$ or $B(\infty)$, if $\mathfrak{g} = B(\infty)$;
- $A(n)$, $A(\infty)$, $C(n)$, or $C(\infty)$ with at most one simple component isomorphic to $C(n)$ or $C(\infty)$, if $\mathfrak{g} = C(\infty)$;
- $A(n)$, $A(\infty)$, $D(n)$, or $D(\infty)$ with at most one simple component isomorphic to $D(n)$ or $D(\infty)$, if $\mathfrak{g} = D(\infty)$.

(ii) If $\mathfrak{p}^{ss} \not\simeq 0$, then $\mathfrak{p}^{ss} + \mathfrak{h} \simeq (\oplus_{t\in T}\mathfrak{k}^t) \oplus Z$, where Z is abelian and $\mathfrak{k}^t$ is isomorphic to $\mathfrak{gl}(n)$ or $\mathfrak{gl}(\infty)$ for any $t \in T$ except at most one index $t_0 \in T$ for which $\mathfrak{k}^{t_0}$ is a root simple direct limit Lie algebra. □

Proof. (i) Let $>_\mathfrak{p}$, as in Proposition 4, be the partial order corresponding to $\mathfrak{p}$. Partition the set $\{\varepsilon_i\}$ (respectively, $\{0\} \cup \{\pm\varepsilon_i\}$) into subsets in such a way that two elements are comparable with respect to $>_\mathfrak{p}$ if and only if they belong to different subsets. Denote by S' the resulting set of subsets of $\{\varepsilon_i\}$ (respectively, of $\{0\} \cup \{\pm\varepsilon_i\}$). Let furthermore S be a subset of S' which contains exactly one element of any pair s, s' of mutually opposite elements of S'; and for every $s \in S$, define $\mathfrak{g}^s$ to be the Lie algebra generated by $\mathfrak{g}^\alpha$, where $\alpha \neq 0$ belongs to s or is a sum of any two elements of s. It is straightforward to verify that $\mathfrak{p}^{ss} \simeq \oplus_{s\in S, \mathfrak{g}^s \neq 0}\mathfrak{g}^s$ is the decomposition of $\mathfrak{p}^{ss}$ into a direct sum of root simple direct limit Lie algebras and that this decomposition satisfies (i).

(ii) The main point is to notice that if $s \neq -s$, then $\mathfrak{g}^s \simeq A(n)$ for some n or $\mathfrak{g}^s \simeq A(\infty)$ and, furthermore, that there is at most one $s \in S$ such that $s = -s$. To complete the proof, it remains to show that each of the Lie subalgebras $\mathfrak{g}^s$ for $s \neq -s$ can be extended to a Lie subalgebra isomorphic to $\mathfrak{gl}(n)$ or $\mathfrak{gl}(\infty)$. This latter argument is purely combinatorial and we leave it to the reader. ■

3 The shadow of an irreducible weight module

Let $\mathfrak{g}$ be a Lie algebra with a root decomposition, let M be an irreducible generalized weight $\mathfrak{g}$-module, and let λ be a fixed point in $\operatorname{supp} M$. For any $\alpha \in \Delta$, consider the set $m^\lambda_\alpha := \{q \in \mathbb{R} \mid \lambda + q\alpha \in \operatorname{supp} M\} \subset \mathbb{R}$. There are four possible types of sets m^λ_α, as follows: bounded in both directions; unbounded in both directions; bounded from above but unbounded from below; unbounded from above but bounded from below. It is proved in [DMP] that, if $\mathfrak{g}$ is finite-dimensional, then the type of m_α^λ depends only on α and not on λ, and therefore the module M itself determines a partition of Δ into four mutually disjoint subsets:

$$\begin{aligned}
\Delta_M^F &:= \left\{\alpha \in \Delta \mid m_\alpha^\lambda \text{ is bounded in both directions}\right\},\\
\Delta_M^I &:= \left\{\alpha \in \Delta \mid m_\alpha^\lambda \text{ is unbounded in both directions}\right\},\\
\Delta_M^+ &:= \left\{\alpha \in \Delta \mid m_\alpha^\lambda \text{ is bounded from above and unbounded from below}\right\},\\
\Delta_M^- &:= \left\{\alpha \in \Delta \mid m_\alpha^\lambda \text{ is bounded from below and unbounded from above}\right\}.
\end{aligned}$$

The corresponding decomposition

$$\mathfrak{g} = \left(\mathfrak{g}_M^F + \mathfrak{g}_M^I\right) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^-, \tag{4}$$

where $\mathfrak{g}_M^F := \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta_M^F} \mathfrak{g}^\alpha)$, $\mathfrak{g}_M^I := \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta_M^I} \mathfrak{g}^\alpha)$, and $\mathfrak{g}_M^\pm := \oplus_{\alpha\in\Delta_M^\pm} \mathfrak{g}^\alpha$, is the M-*decomposition* of $\mathfrak{g}$. The triple $(\mathfrak{g}_M^I, \mathfrak{g}_M^+, \mathfrak{g}_M^-)$ is the *shadow of* M *onto* $\mathfrak{g}$. If $\mathfrak{g}$ is infinite-dimensional, we say that *the shadow of* M *onto* $\mathfrak{g}$ *is well defined* if it is true that the type of m_α^λ depends only on α and not on λ. Then the decomposition (4) (as well as the triple $(\mathfrak{g}_M^I, \mathfrak{g}_M^+, \mathfrak{g}_M^-)$) is well defined.

In the case where $\mathfrak{g}$ is finite-dimensional and reductive, it is furthermore true that $\mathfrak{p}_M := (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra of $\mathfrak{g}$ whose reductive part is $\mathfrak{g}_M^F + \mathfrak{g}_M^I$, and that there is a natural surjection

$$\varphi_M : U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^{\mathfrak{g}_M^+} \to M,$$

where $M^{\mathfrak{g}_M^+}$ is the irreducible $(\mathfrak{g}_M^F + \mathfrak{g}_M^I)$-submodule of M which consists of all vectors in M annihilated by $\mathfrak{g}_M^+$. Moreover, $\operatorname{supp} M$ simply coincides with $\operatorname{supp}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^{\mathfrak{g}_M^+})$. This is the Fernando-Futorny parabolic induction theorem (see [Fe] and also [DMP]), and in particular it provides an explicit description of $\operatorname{supp} M$.

The main purpose of this paper is to understand analogues of these results for a root reductive direct limit Lie algebra $\mathfrak{g}$. Roughly, the situation turns out to be as follows: the shadow of an arbitrary irreducible $\mathfrak{g}$-module M exists and defines a parabolic subalgebra $\mathfrak{p}_M$ of $\mathfrak{g}$; however, the parabolic induction theorem does not hold. The reason for the latter is that, as the nilpotent Lie algebra $\mathfrak{g}_M^+$ is infinite-dimensional, it may not annihilate any nonzero vector in M; such a module M is constructed in Example 5. Nevertheless, the existence of the shadow and the direct limit structure on $\mathfrak{g}$ enable us to obtain an explicit description of $\operatorname{supp} M$ for any M.

Throughout the rest of this paper (except in the Appendix), $\mathfrak{g} := \varinjlim \mathfrak{g}_n$ will be a root reductive direct limit Lie algebra such that all φ_n are embeddings, and M will be a fixed irreducible weight module over $\mathfrak{g}$.

Theorem 2. The shadow of M is well defined. □

Proof. Note first that, for any given $\lambda \in \operatorname{supp} M$, there exist irreducible $(\mathfrak{g}_n + \mathfrak{h})$-modules M_n such that $\lambda \in \operatorname{supp} M_n$ for every n, and $\operatorname{supp} M = \cup_n \operatorname{supp} M_n$. Indeed, fix $m \in M^\lambda$, $m \neq 0$ and define M_n as any irreducible quotient of the $(\mathfrak{g}_n + \mathfrak{h})$-module $U(\mathfrak{g}_n + \mathfrak{h}) \cdot m$. One checks immediately that $\operatorname{supp} M = \cup_n \operatorname{supp} M_n$.

Fix $\alpha \in \Delta$. To prove the theorem, we need to show that if m_α^λ is bounded from above for some $\lambda \in \operatorname{supp} M$ (the proof for the case when m_α^λ is bounded from below is exactly the same), then m_α^μ is bounded from above for any other $\mu \in \operatorname{supp} M$. First, let $\lambda' = \lambda + k\alpha$ be the end point of the α-string through λ in $\operatorname{supp} M$. Then fix $\mu \in \operatorname{supp} M$ and pick N so that $\mu \in \operatorname{supp} M_N$, $\lambda' \in \operatorname{supp} M_N$, and $\alpha \in \Delta_N$. Consider now the α-string through μ. It is the union of the α-strings through μ in $\operatorname{supp} M_n$ for $n = N, N+1, \ldots$. The parabolic induction theorem implies that $\operatorname{supp} M_n$ (and, in particular, the α-string through μ in $\operatorname{supp} M_n$) is contained entirely in an affine half-space in $\lambda + \langle \Delta_n \rangle_{\mathbb{R}}$ whose boundary (affine) hyperplane H_n contains λ' and is spanned by vectors in Δ_n. Furthermore, $H_n' := H_n \cap (\lambda + \langle \Delta_N \rangle_{\mathbb{R}})$ is a hyperplane in $\lambda + \langle \Delta_N \rangle_{\mathbb{R}}$ which contains λ' and is spanned by vectors in Δ_N. Since the α-string through μ belongs to $\lambda + \langle \Delta_N \rangle_{\mathbb{R}}$, the α-string through μ in $\operatorname{supp} M_n$ is bounded by H_n'. But there are only finitely many hyperplanes in $\lambda + \langle \Delta_N \rangle_{\mathbb{R}}$ passing through λ' and spanned by vectors in Δ_N and hence among the hyperplanes H_n' for $n = N, N+1, \ldots$ only finitely many are different. Therefore, the α-strings through μ in $\operatorname{supp} M_n$ are uniformly bounded from above; i.e., m_α^μ is bounded from above. ■

The next theorem describes the structure of the M-decomposition and is an exact analogue of the corresponding theorem for finite-dimensional reductive Lie algebras.

Theorem 3. (i) $\mathfrak{g}_M^F$, $\mathfrak{g}_M^I$, $\mathfrak{g}_M^+$, and $\mathfrak{g}_M^-$ are Lie subalgebras of $\mathfrak{g}$, and $\mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{g}_M^F$ and $\mathfrak{g}_M^I$.

(ii) $[(\mathfrak{g}_M^F)^{ss}, (\mathfrak{g}_M^I)^{ss}] = 0$ and therefore $\mathfrak{g}_M^{FI} := \mathfrak{g}_M^F + \mathfrak{g}_M^I$ is a Lie subalgebra of $\mathfrak{g}$.

(iii) $\mathfrak{g}_M^+$ and $\mathfrak{g}_M^-$ are $\mathfrak{g}_M^{FI}$-modules.

(iv) $\mathfrak{p}_M := \mathfrak{g}_M^{FI} \oplus \mathfrak{g}_M^+$ and $\mathfrak{g}_M^{FI} \oplus \mathfrak{g}_M^-$ are (mutually opposite) parabolic subalgebras of $\mathfrak{g}$. □

Proof. (i) Let $\alpha, \beta \in \Delta$ be such that $\alpha + \beta \in \Delta$. Lemma 2 in [PS] implies that if m_α^λ and m_β^λ are bounded from above, then so is $m_{\alpha+\beta}^\lambda$. Furthermore, noting that $\operatorname{supp} M = \cup_n \operatorname{supp} M_n$ for some irreducible $(\mathfrak{g}_n + \mathfrak{h})$-modules M_n (see the proof of Theorem 2) and that the support of every irreducible weight $(\mathfrak{g}_n + \mathfrak{h})$-module is convex, we conclude that $\operatorname{supp} M$ is convex.

Therefore if m_α^λ and m_β^λ are unbounded from above, so is $m_{\alpha+\beta}^\lambda$. These two facts imply immediately that all four subspaces $\mathfrak{g}_M^F$, $\mathfrak{g}_M^I$, $\mathfrak{g}_M^+$, and $\mathfrak{g}_M^-$ are subalgebras of $\mathfrak{g}$. The fact that $\mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{g}_M^F$ and $\mathfrak{g}_M^I$ is obvious.

(ii) If $\alpha' \in \Delta_M^F$ and $\beta' \in \Delta_M^I$, then $\alpha' + \beta' \notin \Delta$. Indeed, if $\alpha' + \beta' \in \Delta$ and $m_{\alpha'+\beta'}^\lambda$ is bounded from above, then $m_{\beta'}^\lambda$ would be bounded from above because $\beta' = -\alpha' + (\alpha' + \beta')$. If, on the other hand, $\alpha' + \beta' \in \Delta$ and $m_{\alpha'+\beta'}^\lambda$ is unbounded from above, then $m_{\alpha'}^\lambda$ would be unbounded from above because $\alpha' = -\beta' + (\alpha' + \beta')$. Since both of these conclusions contradict the choice of α' and β', we obtain that $\alpha' + \beta' \notin \Delta$ and thus that $[(\mathfrak{g}_M^F)^{ss}, (\mathfrak{g}_M^I)^{ss}] = 0$.

(iii) If $\alpha' \in \Delta_M^F$, $\beta' \in \Delta_M^+$, and $\alpha' + \beta' \in \Delta$, then again $m_{\alpha'+\beta'}^\lambda$ is bounded from above. Assuming that $m_{\alpha'+\beta'}^\lambda$ is bounded from below, we obtain that $m_{\beta'}^\lambda$ is bounded from below as well since $-\beta' = -(\alpha' + \beta') + \alpha'$, which contradicts the fact that $\beta' \in \Delta_M^+$. Hence $m_{\alpha'+\beta'}^\lambda$ is unbounded from below and $\alpha' + \beta' \in \Delta_M^+$. This proves that $\mathfrak{g}_M^+$ is a $\mathfrak{g}_M^F$-module. One shows in a similar way that $\mathfrak{g}_M^\pm$ are $\mathfrak{g}_M^F$- and $\mathfrak{g}_M^I$-modules.

(iv) is a direct corollary of (i), (ii), and (iii). ■

We define an irreducible weight $\mathfrak{g}$-module M to be *cuspidal* if and only if $\mathfrak{g} = \mathfrak{g}_M^I$.

In the rest of this section, we prove that for every parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, there is an irreducible $\mathfrak{g}$-module M such that $\mathfrak{p} = \mathfrak{p}_M$. Indeed, there is the following more general theorem.

Theorem 4. Let $\mathfrak{g}$ be a root reductive direct limit Lie algebra. For any given splitting $\Delta = \Delta^F \sqcup \Delta^I \sqcup \Delta^+ \sqcup \Delta^-$ with $\Delta^F = -\Delta^F$, $\Delta^I = -\Delta^I$, $\Delta^- = -\Delta^+$, and such that its corresponding decomposition

$$\mathfrak{g} = (\mathfrak{g}^F + \mathfrak{g}^I) \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^- \tag{5}$$

satisfies properties (i)–(iii) of Theorem 3, there exists an irreducible weight module M for which (5) is the M-decomposition of $\mathfrak{g}$, i.e., for which $\mathfrak{g}_M^F = \mathfrak{g}^F$, $\mathfrak{g}_M^I = \mathfrak{g}^I$, and $\mathfrak{g}_M^\pm = \mathfrak{g}^\pm$. □

Proof. We start with the observation that it suffices to prove the theorem for a root simple direct limit Lie algebra. Indeed, let $\mathfrak{g} \simeq (\oplus_{s\in S}\mathfrak{g}^s) \Subset A$ be as in Theorem 1. Assume, furthermore, that for each s, M^s is an irreducible weight $\mathfrak{g}^s$-module corresponding (as in the theorem) to the restriction of decomposition (5) to $\mathfrak{g}^s$. Fix nonzero vectors $m^s \in M^s$. Then the reader will verify straightforwardly that, for any pair $(M^{ss} := (\otimes_s M^s)(\otimes_s m^s), \lambda)$ as in Proposition 2(iii), the $\mathfrak{g}$-module $M := M^{ss}(\lambda)$ is as required by the theorem. Therefore, in the rest of the proof, we will assume that $\mathfrak{g}$ is a root simple direct limit Lie algebra. We will prove first that cuspidal modules exist.

Lemma 1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. For any given weight $\mu \in \mathfrak{h}^*$, there exists a cuspidal $\mathfrak{g}$-module M such that the center Z of $U(\mathfrak{g})$ acts on M via the central character $\chi_\mu : Z \to \mathbb{C}$ (obtained by extending μ to a homomorphism $\tilde{\mu} : S^\cdot(\mathfrak{h}) \to \mathbb{C}$ and composing with Harish-Chandra's homomorphism $Z \to S^\cdot(\mathfrak{h})$). □

Proof of Lemma 1. It goes by induction on the rank of $\mathfrak{g}$. It is a classical fact that $sl(2)$ admits a cuspidal module of any given central character, so it remains to make the induction step. Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$ and such that the rank of its semisimple part equals $\operatorname{rk}\mathfrak{g} - 1$. Note that, applying the induction assumption to the semisimple part of $\mathfrak{g}'$ (which clearly is a simple Lie algebra), and then tensoring with an appropriate one-dimensional representation of the center of $\mathfrak{g}'$, we see that the claim of the lemma is also true for $\mathfrak{g}'$. Thus we can fix M' to be a cuspidal $\mathfrak{g}'$-module with central character $\chi'_{\mu'}$, where $\mu' \in \mathfrak{h}^*$ and $w'(\mu') - \mu \notin \langle\Delta\rangle_{\mathbb{Z}}$ for any w' in the Weyl group W' of $\mathfrak{g}'$. Denote by U' the subalgebra of $U(\mathfrak{g})$ generated by $U(\mathfrak{g}')$ and Z. As $U(\mathfrak{g})$ is a free Z-module (Kostant's theorem) and $U(\mathfrak{g}') \cap Z = \mathbb{C}$, U' is isomorphic to $U(\mathfrak{g}') \otimes_{\mathbb{C}} Z$. Therefore, the tensor product $M'_{\chi_\mu} := M' \otimes \mathbb{C}_{\chi_\mu}$, $\mathbb{C}_{\chi_\mu}$ being the one-dimensional Z-module corresponding to χ_μ, is a well-defined U'-module. Consider now any irreducible quotient M of the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U'} M'_{\chi_\mu}$. Obviously, M is a weight module and we claim that M is cuspidal. Assume the contrary. Since M' is a cuspidal $\mathfrak{g}'$-module and the rank of the semisimple part of $\mathfrak{g}'$ is $\operatorname{rk}\mathfrak{g} - 1$, the fact that M is not cuspidal means that M' is a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M''$, where $\mathfrak{p} \supset \mathfrak{g}'$ and M'' is a cuspidal $\mathfrak{g}'$-module of central character χ'_μ. Then, since $U(\mathfrak{g})$ is an integrable $U(\mathfrak{g}')$-module and M' is a subquotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M''$, the central character of M' equals $\chi_{w'(\mu')-\eta}$ for some $\eta \in \langle\Delta\rangle_{\mathbb{Z}}$ and some $w' \in W'$. But as $\chi_{w'(\mu')-\eta} = \chi'_{\mu'}$, we have $w'(\mu') - \eta \in W' \cdot \mu'$; i.e., $w'(\mu') - \mu \in \langle\Delta\rangle_{\mathbb{Z}}$. This contradiction implies that M is cuspidal. ■

If $\mathfrak{g} = \mathfrak{g}^I$, $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ being an infinite-dimensional root simple direct limit Lie algebra, we can assume that $\operatorname{rk}\mathfrak{g}_{n+1} - \operatorname{rk}\mathfrak{g}_n = 1$. We then construct M^I as the direct limit $\varinjlim M_n$, each M_n being a cuspidal $\mathfrak{g}_n$-module built by induction precisely as in the proof of Lemma 1. This proves the theorem in the cuspidal case.

Let now $\mathfrak{g} \neq \mathfrak{g}^I$. According to Theorem 1, $\mathfrak{g}^F \simeq \oplus_{s\in S}\mathfrak{g}^s \Subset A^F$ and $\mathfrak{g}^I \simeq \oplus_{t\in T}\mathfrak{g}^t \Subset A^I$, where each of the algebras $\mathfrak{g}^r$ for $r \in S \cup T$ is a root simple direct limit Lie algebra and A^F and A^I are, respectively, abelian subalgebras of $\mathfrak{g}^F$ and $\mathfrak{g}^I$. Since the theorem is proved for the cuspidal case, we can choose a cuspidal irreducible $\mathfrak{g}^t$ module M^t for each $t \in T$ and fix a nonzero vector $m^t \in M^t$. Then $\widehat{M^I} := (\otimes_{t\in T} M^t)(\otimes_t m^t)$ is a cuspidal $(\mathfrak{g}^I)^{ss}$-module. Let M' denote $\widehat{M^I}$ considered as a $(\mathfrak{g}^F + \mathfrak{g}^I)^{ss}$-module with trivial action of $(\mathfrak{g}^F)^{ss}$.

First, consider the case where $\mathfrak{g}$ is finite-dimensional. Denote by W^I the Weyl group of $(\mathfrak{g}^I)^{ss}$. By Lemma 1, we can assume that the $(\mathfrak{g}^I)^{ss}$-module $\hat{M}^I$ is chosen in such a way that the central characater of $\hat{M}^{I}$ corresponds to an orbit $W^{I}\cdot\eta$ for which the set $W^{I}\cdot\eta-\eta$ does not contain integral weights of $(\mathfrak{g}^{I})^{ss}$. Let M^{FI} denote $\hat{M}^{I}$ with its obvious $(\mathfrak{g}^{F}+\mathfrak{g}^{I})\oplus\mathfrak{g}^{+}$-module structure, and let M be the irreducible quotient of $U(\mathfrak{g})\otimes_{U((\mathfrak{g}^{F}+\mathfrak{g}^{I})\oplus\mathfrak{g}^{+})} M^{FI}$. Then the condition on the central characater of $\hat{M}^{I}$ ensures that the M-decomposition of $\mathfrak{g}$ is nothing but (5). The last statement can be easily verified, which we leave to the reader. Theorem 4 is therefore proved for a finite-dimensional reductive $\mathfrak{g}$.

A direct generalization of this argument does not go through for an infinite-dimensional $\mathfrak{g}$. Instead, there is the following lemma which allows us to avoid referring to central characters.

Lemma 2. If $\mathfrak{g}$ is infinite-dimensional, then the $(\mathfrak{g}^{F}+\mathfrak{g}^{I})^{ss}$-module structure on M' can be extended to a $(\mathfrak{g}^{F}+\mathfrak{g}^{I})$-module structure in such a way that $\langle\mu',\alpha\rangle\notin\mathbb{Z}$ for some (and hence any) $\mu'\in\operatorname{supp}M'$ and any $\alpha\in\Delta^{+}\sqcup\Delta^{-}$. $\square$

Proof of Lemma 2. Theorem 3(iv) implies that $\mathfrak{g}_{M}^{F}+\mathfrak{g}_{M}^{I}=(\mathfrak{p}_{M})^{ss}+\mathfrak{h}$. We will present the proof in the case where $\mathfrak{g}_{M}^{F}+\mathfrak{g}_{M}^{I}\neq\mathfrak{h}$. The case where $\mathfrak{g}_{M}^{F}+\mathfrak{g}_{M}^{I}=\mathfrak{h}$ is dealt with in a similar way.

Using Proposition 5(ii), we conclude that $\mathfrak{g}_{M}^{F}+\mathfrak{g}_{M}^{I}\neq\mathfrak{h}$ implies

$$\mathfrak{g}_{M}^{F}+\mathfrak{g}_{M}^{I}\simeq(\oplus_{r\in R}\mathfrak{g}^{r})\oplus\mathfrak{g}^{0}\oplus Z, \tag{6}$$

where each $\mathfrak{g}^{r}$ is isomorphic to $gl(n)$ for some n or to $gl(\infty)$; R is a finite or countable set which does not contain 0 (and may be empty); $\mathfrak{g}^{0}$ is a root simple direct limit Lie algebra; and Z is abelian.

Let Δ^{s} be the root system of $\mathfrak{g}^{s}$ for $s\in R\cup\{0\}$. Set ${}^{s}\mathbb{Z}:=\{k\in(\mathbb{Z}_{+}\backslash\{0\})\mid$ there is $l\in(\mathbb{Z}_{+}\backslash\{0\})$ with $\varepsilon_{k}-\varepsilon_{l}\in\Delta^{s}\}$ and ${}^{\#}\mathbb{Z}:=(\mathbb{Z}_{+}\backslash\{0\})\backslash(\cup_{s\in R\cup\{0\}}{}^{s}\mathbb{Z})$. As explained in Section 1, any sequence $\{\lambda^{n}\}_{n=1,2,\ldots}\subset\mathbb{C}$ determines a weight λ of $(\mathfrak{g}^{F}+\mathfrak{g}^{I})^{ss}$; however, λ reconstructs only the subsequence $\{\lambda^{n}\}_{n\in\cup_{s\in R\cup\{0\}}{}^{s}\mathbb{Z}}$ up to an additive constant for every index from R. Fix now $\mu\in\operatorname{supp}M'$ and let $\{\mu^{n}\}_{n=1,2,\ldots}$ be a sequence which determines μ. To prove the lemma, it suffices to find constants $c^{i}\in\mathbb{C}$ for $i\in\mathbb{Z}$ with the following properties:

- $c^{k}=c^{l}$ whenever k and l belong to one and the same set ${}^{s}\mathbb{Z}$;
- $c^{k}=0$ for $k\in{}^{0}\mathbb{Z}$;
- $\mu^{k}+c^{k}\notin\mathbb{Z}$ for every $k\in(\mathbb{Z}_{+}\backslash\{0\})$ for which $\varepsilon_{k}\in\Delta^{+}\sqcup\Delta^{-}$; $2(\mu^{k}+c^{k})\notin\mathbb{Z}$ for every $k\in(\mathbb{Z}_{+}\backslash\{0\})$ for which $2\varepsilon_{k}\in\Delta^{+}\sqcup\Delta^{-}$; and $\pm(\mu^{k}+c^{k})\pm(\mu^{l}+c^{l})\notin\mathbb{Z}$ for every $k,l\in(\mathbb{Z}_{+}\backslash\{0\})$ for which $\pm\varepsilon_{k}\pm\varepsilon_{l}\in\Delta^{+}\sqcup\Delta^{-}$.

Then μ' will be the weight of $(\mathfrak{g}^{F}+\mathfrak{g}^{I})$ determined by the sequence $\{\mu^{n}+c^{n}\}_{n=1,2,\ldots}$.

Consider the set of sequences $\{c^{k}\}_{k\in P\subset(\mathbb{Z}_{+}\backslash\{0\})}$ with the three properties as above. Introduce an order $<$ on this set by putting $\{c'^{k'}\}_{k'\in P'}<\{c''^{k''}\}_{k''\in P''}$ if and only if $P'\subset P''$

and $c'^k = c''^k$ for every $k \in P'$. The reader will check that every chain (with respect to $<$) of sequences is bounded and that for every $\{c^k\}_{k\in P}$ with $P \neq (\mathbb{Z}_+\backslash\{0\})$, there is a sequence greater than $\{c^k\}_{k\in P}$. Therefore any maximal element, which exists by Zorn's lemma, is a sequence $\{c^k\}$ with the required properties. Lemma 2 is proved. ■

Assuming that $\mathfrak{g}$ is infinite-dimensional, let now M^{FI} be M' considered as a $(\mathfrak{g}^F + \mathfrak{g}^I) \oplus \mathfrak{g}^+$-module with trivial action of $\mathfrak{g}^+$. Then we define M as the (unique) irreducible quotient of the $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U((\mathfrak{g}^F+\mathfrak{g}^I)\oplus\mathfrak{g}^+)} M^{FI}$. Using Lemma 2, the reader will verify that the M-decomposition of $\mathfrak{g}$ is precisely the decomposition (5). The proof of Theorem 4 is therefore complete. ■

4 Integrable modules

Proposition 6. Let M be integrable. Then we have the following.

(i) $\mathfrak{g} = \mathfrak{g}^F_M + \mathfrak{g}^I_M$; furthermore, $\mathfrak{g} \neq \mathfrak{g}^F_M$ implies $\dim M^\lambda = \infty$ for any $\lambda \in \operatorname{supp} M$.

(ii) If $\mathfrak{g}$ is infinite-dimensional and simple, M is either cuspidal or $\mathfrak{g} = \mathfrak{g}^F_M$, and both cases are possible. □

Proof. (i) As a consequence of the integrability of M, $\operatorname{supp} M$ is W-invariant. Thus $\Delta^\pm_M = \emptyset$ and $\mathfrak{g} = \mathfrak{g}^F_M + \mathfrak{g}^I_M$. Furthermore, for any $\alpha \in \Delta$, the subalgebra $\mathfrak{g}^{\mathbb{R}\alpha}$ of $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ is isomorphic to $sl(2)$, and the integrability of M implies that as a $\mathfrak{g}^{\mathbb{R}\alpha}$- module, M is isomorphic to a direct sum of finite-dimensional modules. The assumption that $\dim M^\lambda < \infty$ for some $\lambda \in \operatorname{supp} M$ would lead us to the conclusion that λ belongs to the support of only finitely many of the finite-dimensional $\mathfrak{g}^{\mathbb{R}\alpha}$-modules, which would mean that $\operatorname{supp} M$ is finite in the direction of both α and $-\alpha$. Therefore $\mathfrak{g} \neq \mathfrak{g}^F_M$ implies $\dim M^\lambda = \infty$ for any $\lambda \in \operatorname{supp} M$.

(ii) If both $\mathfrak{g}^F_M \neq \mathfrak{h}$ and $\mathfrak{g}^I_M \neq \mathfrak{h}$, Theorems 1 and 3 imply that $(\mathfrak{g}^F_M)^{ss}$ and $(\mathfrak{g}^I_M)^{ss}$ are proper ideals in $\mathfrak{g}$, which is a contradiction. Thus $\mathfrak{g}^F_M = \mathfrak{h}$ or $\mathfrak{g}^I_M = \mathfrak{h}$; i.e., $\mathfrak{g} = \mathfrak{g}^I_M$ or $\mathfrak{g} = \mathfrak{g}^F_M$.

Clearly, $\mathfrak{g} = \mathfrak{g}^F_M$ if $M = \mathfrak{g}$ is the adjoint representation, so the case $\mathfrak{g} = \mathfrak{g}^F_M$ is obviously possible. To prove that integrable cuspidal modules exist, it is enough to construct a tower of embeddings of irreducible finite-dimensional $\mathfrak{g}_n$-modules M_n, $\ldots \to M_n \to M_{n+1} \to \ldots$, such that the support of M_n is shorter than the support of M_{n+1} in all root directions of $\mathfrak{g}_n$. Obviously, $M := \varinjlim M_n$ is then an integrable cuspidal $\mathfrak{g}$-module. Here is an explicit example for $\mathfrak{g} = A(\infty)$. Let $\lambda_n = n\varepsilon_1 - n\varepsilon_2$ and let $\mathfrak{b}_n$ be the Borel subalgebra of $\mathfrak{g}_n$ corresponding to the order $\varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_{n+1} < \varepsilon_1$. Set $M_n := V_{\mathfrak{b}_n}(\lambda_n)$ for $n \geq 2$. There is a unique (up to a scalar multiple) embedding of $\mathfrak{g}_n$-modules $M_n \to M_{n+1}$ (this can be verified directly or by using the classical branching rule (see, for instance, Ch.4, §6 in [Zh])); and furthermore, the support of M_{n+1} is seen immediately to be longer than the support of M_n in each root direction of $\mathfrak{g}_n$. The proof of Proposition 6 is therefore complete. ■

Obviously, if M is integrable and is a highest weight module with respect to some Borel subalgebra $\mathfrak{b}$, then $\mathfrak{g} = \mathfrak{g}^F_M$. The existence of cuspidal integrable modules is a new phenomenon. It is different from the case of a finite-dimensional Lie algebra, where any integrable irreducible module is finite-dimensional and is thus a highest weight module with respect to every Borel subalgebra. In contrast, for $\mathfrak{g} = A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$, the trivial $\mathfrak{g}$-module is the only irreducible $\mathfrak{g}$-module that is a highest weight module for all Borel subalgebras. Moreover, as we will see in Example 3 below, the equality $\mathfrak{g} = \mathfrak{g}^F_M$ does not guarantee the existence of a Borel subalgebra with respect to which M is a highest weight module. We define M to be *finite integrable* if and only if $\mathfrak{g} = \mathfrak{g}^F_M$. The rest of this section is devoted to the study of finite integrable irreducible $\mathfrak{g}$-modules M over an arbitrary root reductive direct limit Lie algebra $\mathfrak{g}$.

The simplest type of finite integrable modules are highest weight integrable modules, and they are studied in the recent papers [BB], [NRW], and [Ne]. In [BB], the highest weights of integrable highest weight modules are computed explicitly. In [NRW], the integrable modules $V_{\mathfrak{b}}(\lambda)$ appear in the context of Borel-Weil-Bott's theorem and are realized as the unique nonzero cohomology groups of line bundles on G/B. In [Ne], irreducible highest weight modules with respect to general Borel subalgebras are considered; furthermore, it is proved that their integrability is equivalent to unitarizability. In Theorem 5 below, we discuss highest weight modules and, in particular, establish a direct limit version of H. Weyl's character formula for integrable highest weight modules.

First we need to recall the notion of basis of a Borel subalgebra of $\mathfrak{g}$. A subset $\Sigma \subset \Delta^+$ is a *basis* of $\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta^+}\mathfrak{g}^\alpha)$ if and only if Σ is a linearly independent set and every element of Δ^+ is a linear combination of elements of Σ with nonnegative integer coefficients; the elements of Σ are then the *simple roots* of $\mathfrak{b}$. Not every Borel subalgebra admits a basis. For root simple direct limit Lie algebras, a basis of $\mathfrak{b}$ is the same as a weak basis in the terminology of [DP2]; and in [DP2], all Borel subalgebras admitting a weak basis are described. The result of [DP2] implies that $\mathfrak{b}$ admits a basis if and only if the corresponding order on $\{\varepsilon_i\}$, or, respectively, on $\{0\} \cup \{\pm\varepsilon_i\}$, has the following property: for every pair of elements of $\{\varepsilon_i\}$ (respectively, of $\{0\} \cup \{\pm\varepsilon_i\}$), there are only finitely many elements between them. This latter criterion has been established also by K.-H. Neeb in [Ne].

Theorem 5. Let $M = V_{\mathfrak{b}}(\lambda)$.

(i) M is integrable if and only if λ is an integral $\mathfrak{b}$-dominant weight. Furthermore, if M is integrable, then $\operatorname{supp} M = C^\lambda$, where C^λ is the intersection of the convex hull of

$W \cdot \lambda$ with $\lambda + \langle \Delta \rangle_{\mathbb{Z}}$.

(ii) $M \simeq V_{\mathfrak{b}'}(\lambda')$ for given $\mathfrak{b}'$ and λ' if and only if there exists $w \in W$ for which $\lambda' = w(\lambda)$ and there is a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing both $w(\mathfrak{b})$ and $\mathfrak{b}'$ such that the $\mathfrak{p}$-submodule of M generated by $M^{\lambda'}$ is one-dimensional.

(iii) $\dim M^{\mu} < \infty$ for all $\mu \in \operatorname{supp} M$ if and only if $M \simeq V_{\tilde{\mathfrak{b}}}(\lambda)$ for some Borel subalgebra $\tilde{\mathfrak{b}}$ of $\mathfrak{g}$ which admits a basis.

(iv) If M is integrable, $\mathfrak{b}$ admits a basis, and $\operatorname{ch} M := \sum_{\mu \in \operatorname{supp} M} \dim M^{\mu} \cdot e^{\mu}$ is the formal character of M, we have

$$D \cdot \operatorname{ch} M = \sum_{w \in W} (\operatorname{sgn} w) e^{w(\lambda + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}}}, \tag{7}$$

where $D = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$ and $\rho_{\mathfrak{b}} \in \mathfrak{h}^*$ is a weight for which $\rho_{\mathfrak{b}}(\alpha) = 1$ for all simple roots of $\mathfrak{b}$. □

Proof. (i) The proof is an exercise. In [BB], a more general criterion for the integrability of $V_{\mathfrak{b}}(\lambda)$ is established for simple direct limit Lie algebras which are not necessarily root simple direct limit Lie algebras.

(ii) Define $\mathfrak{p}_{\lambda} \supset \mathfrak{b}$ as the maximal parabolic subalgebra of $\mathfrak{g}$ such that the $\mathfrak{p}_{\lambda}$-submodule of M generated by M^{λ} is one-dimensional. Then $M \simeq V_{\mathfrak{b}''}(\lambda)$ if and only if $\mathfrak{b}''$ is a subalgebra of $\mathfrak{p}_{\lambda}$. Finally, it is an exercise to check that $M \simeq V_{\mathfrak{b}'}(\lambda')$ if and only if $\lambda' = w(\lambda)$, and $\mathfrak{b}'$ is a subalgebra of $\mathfrak{p}_{\lambda'} = w(\mathfrak{p}_{\lambda})$ for some $w \in W$.

(iii) If $\tilde{\mathfrak{b}}$ admits a basis, then, using the Poincaré-Birkhoff-Witt theorem, one verifies that all weight spaces of the Verma module $\tilde{V}_{\mathfrak{b}}(\lambda)$ are finite-dimensional (for any $\lambda \in \mathfrak{h}^*$), and hence all weight spaces of $V_{\mathfrak{b}}(\lambda)$ are finite-dimensional as well.

Conversely, let $M = V_{\mathfrak{b}}(\lambda)$ be a highest weight module with finite-dimensional weight spaces. We need to prove the existence of $\tilde{\mathfrak{b}}$ which admits a basis and such that $M \simeq V_{\tilde{\mathfrak{b}}}(\lambda)$. Let $\mathfrak{p}_{\lambda}$ be as above and let $\mathfrak{g} \simeq (\oplus_{s \in S} \mathfrak{g}^s) \in\!\!\!\!\!\subset A$ as in Theorem 1. Clearly, there is a Borel subalgebra $\tilde{\mathfrak{b}} \subset \mathfrak{p}_{\lambda}$ that admits a basis if and only if for every $s \in S$, there is a Borel subalgebra $\tilde{\mathfrak{b}^s}$ of $\mathfrak{g}^s$ admitting a basis such that $\tilde{\mathfrak{b}^s} \subset \mathfrak{p}^s_{\lambda} := \mathfrak{p}_{\lambda} \cap \mathfrak{g}^s$. Furthermore, $M^s := V_{\mathfrak{b}^s}(\lambda|_{\mathfrak{h}^s})$ (where $\mathfrak{b}^s := \mathfrak{b} \cap \mathfrak{g}^s$ and $\mathfrak{h}^s := \mathfrak{h} \cap \mathfrak{g}^s$) is an irreducible $\mathfrak{g}^s$-submodule of M. Thus it suffices to prove (iii) for the root simple direct limit Lie algebras $\mathfrak{g}^s$ and their highest weight modules M^s. For a finite-dimensional $\mathfrak{g}^s$, (iii) is trivial, so we need to consider only the case where $\mathfrak{g}^s = A(\infty), B(\infty), C(\infty), D(\infty)$.

Let $>_{\mathfrak{p}^s_{\lambda}}$ be the partial order corresponding to $\mathfrak{p}^s_{\lambda}$ and let $\alpha \in (\Delta^s)^+$ (Δ^s being the root system of $\mathfrak{g}^s$) be a difference of two elements δ_1 and δ_2 of $\{\varepsilon_i\}$ (respectively, of $\{0\} \cup \{\pm\varepsilon_i\}$). Then it is not difficult to check that, if there are infinitely many elements of $\{\varepsilon_i\}$ (respectively, of $\{0\} \cup \{\pm\varepsilon_i\}$) between δ_1 and δ_2 with respect to $>_{\mathfrak{p}^s_{\lambda}}$, the weight space of M^s with weight $\lambda - \alpha$ is infinite-dimensional. Therefore, the assumption that all Borel subalgebras $\tilde{\mathfrak{b}}^s$, such that $M^s \simeq V_{\tilde{\mathfrak{b}}^s}(\lambda|_{\mathfrak{h}^s})$ admit no basis, is contradictory; i.e., a Borel subalgebra $\tilde{\mathfrak{b}}^s$ exists, as required.

(iv) To prove formula (7), it suffices to notice that if $\mathfrak{b}$ admits a basis, then $\mathfrak{g}$ can be represented as $\mathfrak{g} = \varinjlim \mathfrak{g}_n$, where each of the simple roots of $\mathfrak{b}_n = \mathfrak{b} \cap \mathfrak{g}_n$ is a simple root of $\mathfrak{b}_{n+1} = \mathfrak{b} \cap \mathfrak{g}_{n+1}$. Then $M = \varinjlim V_{\mathfrak{b}_n}(\lambda|_{\mathfrak{h}_n})$ and, in both sides of (7), terms of the form $c \cdot e^{\lambda+\mu}$ for $\mu \in \langle \Delta_n \rangle_{\mathbb{R}}$ appear only as they appear in respective sides of (7) for the $\mathfrak{g}_n$-module $V_{\mathfrak{b}\cap\mathfrak{g}_n}(\lambda|_{\mathfrak{h}_n})$. Therefore, Weyl's original formula implies the infinite version (7). ■

We now turn our attention to general finite integrable modules M. Here is an example of a finite integrable M which is not a highest weight module with respect to any Borel subalgebra of $\mathfrak{g}$.

Example 3. Let $\mathfrak{g} = A(\infty)$ and let $\mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to the order $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$. Set $\lambda_n := \varepsilon_1 + \cdots + \varepsilon_n - n\varepsilon_{n+1}$. Since $\lambda_n \in \operatorname{supp} V_{\mathfrak{b}_{n+1}}(\lambda_{n+1})$ and the weight space $V_{\mathfrak{b}_{n+1}}(\lambda_{n+1})^{\lambda_n}$ is one-dimensional, there is a unique (up to a multiplicative constant) embedding of weight $\mathfrak{g}_n$-modules $V_{\mathfrak{b}_n}(\lambda_n) \to V_{\mathfrak{b}_{n+1}}(\lambda_{n+1})$. Set $M := \varinjlim V_{\mathfrak{b}_n}(\lambda_n)$. Then $\mathfrak{g} = \mathfrak{g}_M^F$ and all weight spaces of M are one-dimensional. M is not a highest weight module with respect to any Borel subalgebra of $\mathfrak{g}$, as no weight of M is a highest weight of $V_{\mathfrak{b}_n}(\lambda_n)$ for two consecutive n.

The following two theorems provide an explicit parameterization of all finite integrable modules as well as an explicit description of their supports. Let $W_n \subset W$ denote the Weyl group of $\mathfrak{g}_n$.

Theorem 6. Let M be finite integrable.

(i) $M \simeq \varinjlim M_n$ for some direct system of finite-dimensional irreducible $\mathfrak{g}_n$-modules M_n.

(ii) $\operatorname{supp} M$ determines M up to isomorphism.

(iii) Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Then $M \simeq \varinjlim V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$ for some sequence $\{\lambda_n\}$ of integral weights of $\mathfrak{g}$ such that $\lambda_n|_{\mathfrak{h}_n}$ is a $\mathfrak{b}_n$-dominant weight of $\mathfrak{g}_n$ and λ_n belongs to an edge of the convex hull of $W_{n+1} \cdot \lambda_{n+1}$. □

To prove the theorem, we need a lemma which is a weak version of the classical branching rule and for which we did not find a convenient quotation.

Lemma 3. Let $\mathfrak{k} \subset \mathfrak{l}$ be finite-dimensional reductive Lie algebras whose Cartan subalgebras coincide. Let furthermore N be an irreducible finite-dimensional $\mathfrak{l}$-module with a decomposition $N = \oplus_i N_i$ into irreducible $\mathfrak{k}$-modules. Fix $\nu \in \operatorname{supp} N$. Then the following hold.

(i) There is at least one j_0 such that $\nu \in \operatorname{supp} N_{j_0}$, and $\operatorname{supp} N_{j_0}$ contains $\operatorname{supp} N_j$ for all j with $\nu \in \operatorname{supp} N_j$;

(ii) if $\operatorname{rk}\mathfrak{l}' - \operatorname{rk}\mathfrak{k}' = 1$, $\mathfrak{l}'$ and $\mathfrak{k}'$ being the semisimple parts of $\mathfrak{l}$ and $\mathfrak{k}$, respectively, then $\dim N^{\nu'} = 1$ for any highest weight ν' of N_{j_0}. □

Proof. (i) It is enough to prove (i) in the case where $\operatorname{rk}\mathfrak{l}' - \operatorname{rk}\mathfrak{k}' = 1$, and then apply a simple induction argument. Let $\operatorname{rk}\mathfrak{l}' - \operatorname{rk}\mathfrak{k}' = 1$. Denote by $\Gamma_1, \ldots, \Gamma_p$ the $W_{\mathfrak{k}}$-orbits of a highest weight of N ($W_{\mathfrak{k}}$ being the Weyl group of $\mathfrak{k}$). We can assume that $\Gamma_1, \ldots, \Gamma_p$ are ordered in a way that the convex hull G_i of $\Gamma_{i-1} \cup \Gamma_i$ does not intersect with Γ_j for $j \neq i-1, i$. G_i is a polyhedron with vertices $\Gamma_{i-1} \cup \Gamma_i$, and moreover the convex hull of $\operatorname{supp} N$ equals the union $\cup_{i=2}^{p} G_i$. The edges of G_i are of three types: the edges of the convex hull of Γ_{i-1}; the edges of the convex hull of Γ_i; and the line segments connecting the $\mathfrak{b}_{\mathfrak{k}}$-highest weight among Γ_{i-1} with the $\mathfrak{b}_{\mathfrak{k}}$-highest weight among Γ_i, where $\mathfrak{b}_{\mathfrak{k}}$ is any Borel subalgebra of $\mathfrak{k}$. If ν' is an integral weight which belongs to one of the latter edges, then $\operatorname{supp} V_{\mathfrak{b}_{\mathfrak{k}}}(\nu') = (\nu' + \langle\Delta_{\mathfrak{k}}\rangle_{\mathbb{Z}}) \cap \operatorname{supp} N$. This completes the proof of (i).

(ii) Since ν' belongs to an edge of the convex hull of $\operatorname{supp} N$, we have $\dim N^{\nu'} = 1$. ■

Proof of Theorem 6. (i) Fix $\lambda \in \operatorname{supp} M$. We claim that there is a unique (up to isomorphism) irreducible finite-dimensional $\mathfrak{g}_n + \mathfrak{h}$-module M_n^{λ} such that $\operatorname{supp} M_n^{\lambda} = (\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M$. To check this, it is enough to verify that $((\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M)|_{\mathfrak{h}_n}$ equals the support of some irreducible finite-dimensional $\mathfrak{g}_n$-module. Let M_n' be an irreducible $\mathfrak{g}_n + \mathfrak{h}$-module such that $\lambda \in \operatorname{supp} M_n'$ and $\cup_n \operatorname{supp} M_n' = \operatorname{supp} M$ (such modules are constructed in the proof of Theorem 2). The equality $\mathfrak{g} = \mathfrak{g}_M^F$ implies that $(\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M$ is a finite set, and hence there is n' such that $(\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M$ is contained in $\operatorname{supp} M_{n'}'$. Moreover, $(\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M_{n'}' = (\lambda + \langle\Delta_n\rangle_{\mathbb{Z}}) \cap \operatorname{supp} M$. Applying Lemma 3(i) to the pair of Lie algebras $\mathfrak{g}_n + \mathfrak{h}_{n'} \subset \mathfrak{g}_{n'}$ and the $\mathfrak{g}_{n'}$-module $M_{n'}'$, we define M_n^{λ} as the $\mathfrak{g}_n + \mathfrak{h}$-module obtained by extending a $\mathfrak{g}_n + \mathfrak{h}_{n'}$-module N_{j_0} to a $\mathfrak{g}_n + \mathfrak{h}$-module for which $\lambda \in \operatorname{supp} M_n^{\lambda}$. Lemma 3(ii) implies that there is a unique (up to a scalar multiple) embedding $M_n^{\lambda} \to M_{n+1}^{\lambda}$, and hence the $\mathfrak{g}$-module $M' := \varinjlim M_n^{\lambda}$ is well defined. Noting that M_n^{λ} can be embedded as a submodule of the $\mathfrak{g}_n + \mathfrak{h}$-module M, we conclude that $M \simeq M'$.

(ii) The crucial point is that (according to its construction) M_n^{λ} depends only on $\operatorname{supp} M$ and $\lambda \in \operatorname{supp} M$. Furthermore, for any $\lambda' \in \operatorname{supp} M$, there is m such that $\lambda' \in \operatorname{supp} M_m^{\lambda}$ and hence $M_n^{\lambda} \simeq M_n^{\lambda'}$ for $n > m$. Letting now n go to ∞, and noting that for any pair $\lambda, \lambda' \in \operatorname{supp} M$, there is a compatible system of isomorphisms $M_n^{\lambda} \simeq M_n^{\lambda'}$ for all $n > m$, we conclude that $M = \varinjlim M_n^{\lambda} \simeq \varinjlim M_n^{\lambda'}$, i.e., that $\operatorname{supp} M$ determines M up to isomorphism.

(iii) Since the module M_n^{λ} defined in (i) is finite-dimensional, $M_n^{\lambda} \simeq V_{\mathfrak{b}_n}(\widetilde{\lambda_n})$ for some $\widetilde{\lambda_n} \in \mathfrak{h}_n^*$. There is a unique weight μ of $\mathfrak{g}$ such that $\mu \in \langle\Delta_n\rangle_{\mathbb{R}}$ and $\mu|_{\mathfrak{h}_n} = \lambda|_{\mathfrak{h}_n} - \widetilde{\lambda_n}$. Set

$\lambda_n := \lambda - \mu \in \mathfrak{h}^*$. Then $M \simeq \varinjlim M_n^\lambda \simeq V_{\mathfrak{b}_n}(\widetilde{\lambda_n}) = \varinjlim V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$ and, according to Lemma 3, λ_n belongs to an edge of the convex hull of $W_{n+1} \cdot \lambda_{n+1}$. ■

Theorem 7. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Let $\{\lambda_n\} \subset \mathfrak{h}^*$ be a sequence of integral weights of $\mathfrak{g}$ such that $\lambda_n|_{\mathfrak{b}_n}$ is a $\mathfrak{b}_n$-dominant weight of $\mathfrak{g}_n$ and λ_n belongs to an edge of the convex hull of $W_{n+1} \cdot \lambda_{n+1}$. Then, by the same argument as in Example 3, $M := \varinjlim V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$ is a well-defined irreducible weight $\mathfrak{g}$-module.

(i) M is finite integrable.

(ii) $\operatorname{supp} M = \cup_n C^{\lambda_n}$.

(iii) M is a highest weight module with respect to some Borel subalgebra of $\mathfrak{g}$ if and only if there is n_0 so that $\lambda_n \in W \cdot \lambda_{n_0}$ for any $n \geq n_0$.

(iv) If $M' \simeq \varinjlim V_{\mathfrak{b}_n}(\lambda'_n|_{\mathfrak{h}_n})$, $\{\lambda'_n\}$ being a sequence of integral weights of $\mathfrak{g}$ such that $\lambda'_n|_{\mathfrak{h}_n}$ is a $\mathfrak{b}_n$-dominant weight of $\mathfrak{g}_n$ and λ'_n belongs to an edge of the convex hull of $W_{n+1} \cdot \lambda'_{n+1}$, then $M \simeq M'$ if and only if there is n_0 so that $\lambda_n = \lambda'_n$ for $n \geq n_0$. □

Proof. The proof is not difficult and is left to the reader. ■

Remark. The following question is quite natural and intriguing: How large is the set of finite integrable modules which are not highest weight modules? A rigorous answer would require a way to measure this set. One possible approach is to estimate the number of weights λ_{n+1} which can extend a given sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$. Based on Theorem 7, we can show, using a direct combinatorial approach, that (when $\lambda_1 \neq 0$) any given finite sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ as in Theorem 7 can be extended to an infinite such sequence. Furthermore, if we require that $\operatorname{supp} M \subset \langle\Delta\rangle_{\mathbb{R}}$ (or equivalently, that $\lambda_1 \in \langle\Delta\rangle_{\mathbb{R}}$) at each step, there are only finitely many possibilities for the choice of the next weight in the sequence. A complete combinatorial description of all possible sequences $\{\lambda_n\}$ is an interesting open problem.

Example 4. Let $\mathfrak{g} = A(\infty)$ and $M = \mathfrak{g}$ be the adjoint module. If $\mathfrak{b}$ is the Borel subalgebra corresponding to the order $\varepsilon_1 > \varepsilon_2 > \ldots$, then $M \simeq \varinjlim V_{\mathfrak{b}_n}(\varepsilon_1 - \varepsilon_{n+1})$. Since $W \cdot (\varepsilon_1 - \varepsilon_{n+1}) = \Delta$, Theorem 7(iii) implies that there exists a Borel subalgebra $\mathfrak{b}$ such that the adjoint module is a $\mathfrak{b}$-highest weight module. Furthermore, it is not difficult to verify that all such Borel subalgebras $\mathfrak{b}$ are precisely the Borel subalgebras which correspond to orders on $\{\varepsilon_i\}$ for which there exists a pair of indices i_0, j_0 so that $\varepsilon_{i_0} > \varepsilon_i$ and $\varepsilon_i > \varepsilon_{j_0}$ for all $i \neq i_0$, $i \neq j_0$.

5 Pseudo highest weight modules

In the case of a finite-dimensional Lie algebra, an irreducible weight module M with $\mathfrak{g}_M^I = \mathfrak{h}$ is necessarily a highest weight module for some Borel subalgebra (see [DMP]).

As we already know (Example 3), this is no longer true for the direct limit algebras we consider. We define a *pseudo highest weight module* as an irreducible weight module M such that $\mathfrak{g}^I_M = \mathfrak{h}$. Pseudo highest weight modules provide counterexamples also to the obvious extension of the parabolic induction theorem to root reductive direct limit Lie algebras. (The module M from Example 3 above does not provide such a counterexample since in this case $\mathfrak{p}_M = \mathfrak{g}$ and $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M$.) Indeed, it suffices to construct M with $\mathfrak{g}^F_M = \mathfrak{g}^I_M = \mathfrak{h}$ such that M is not a highest weight module with respect to $\mathfrak{h} \oplus \mathfrak{g}^+_M$ and therefore admits no surjection of the form $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M' \to M$ for an irreducible $\mathfrak{p}_M$-module M'. Here is such an example.

Example 5. Let $\mathfrak{g} = A(\infty)$, $B(\infty)$, $C(\infty)$, or $D(\infty)$. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and let $\lambda \in \mathfrak{h}^*$ be such that $\langle \lambda, \alpha \rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta$. Construct $\{\lambda_n\}$ inductively by setting $\lambda_3 := \lambda$, $\lambda_{n+1} := \lambda_n + \alpha_n$, where α_n is a simple root of $\mathfrak{b}_{n+1}$ which is not a root of $\mathfrak{b}_n$. Since $\dim V_{\mathfrak{b}_{n+1}}(\lambda_{n+1}|_{\mathfrak{h}_{n+1}})^{\lambda_n|_{\mathfrak{h}_n}} = 1$, there is a unique (up to a constant) embedding $V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n}) \to V_{\mathfrak{b}_{n+1}}(\lambda_{n+1}|_{\mathfrak{h}_{n+1}})$, so we can set $M := \varinjlim V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^+_M \oplus \mathfrak{g}^-_M$, where $\mathfrak{h} \oplus \mathfrak{g}^+_M = \mathfrak{b}$. Therefore the only Borel subalgebra with respect to which M could be a highest weight module is $\mathfrak{b}$. However, M has no nonzero $\mathfrak{b}$-highest weight vector. Indeed, such a highest weight vector would be also a highest weight vector of $V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$ for every $n \geq n_0$ while $V_{\mathfrak{b}_n}(\lambda_n|_{\mathfrak{h}_n})$ do not admit a common highest weight vector for two consecutive n. Therefore, M is not a highest weight module. ∎

For any irreducible weight module M, we have the following natural question: What are the integrable root directions of M? That is, for which roots $\alpha \in \Delta$ is M an integrable $\mathfrak{g}^\alpha$-module? It is a remarkable fact that if Δ^{int}_M is the set of all roots α for which M is $\mathfrak{g}^\alpha$-integrable, then $\mathfrak{p}^{\mathrm{int}}_M := \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^{\mathrm{int}}_M} \mathfrak{g}^\alpha)$ is a Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{p}^{\mathrm{int}}_M$ is nothing but the subset of elements in $\mathfrak{g}$ which act locally finitely on M. For a finite-dimensional Lie algebra, this can be proved using Gabber's theorem, [G]; see Corollary 2.7 in [Fe], or Proposition 1 in [PS]. For a root direct limit Lie algebra, the statement follows immediately from the case of a finite-dimensional Lie algebra.

Proposition 7. (i) $\mathfrak{p}^{\mathrm{int}}_M = (\mathfrak{g}^F_M + (\mathfrak{g}^I_M \cap \mathfrak{p}^{\mathrm{int}}_M)) \oplus \mathfrak{g}^+_M$.

(ii) If M is a pseudo highest weight module, then $\mathfrak{p}^{\mathrm{int}}_M = \mathfrak{p}_M$; in particular, $\mathfrak{p}^{\mathrm{int}}_M$ is a parabolic subalgebra of $\mathfrak{g}$. □

Proof. (i) It is obvious that $\mathfrak{g}^F_M \subset \mathfrak{p}^{\mathrm{int}}_M$ and $\mathfrak{g}^+_M \subset \mathfrak{p}^{\mathrm{int}}_M$. Furthermore, $\mathfrak{p}^{\mathrm{int}}_M \cap \mathfrak{g}^-_M = 0$. Indeed, assuming that $\mathfrak{g}^-_M \cap \mathfrak{p}^{\mathrm{int}}_M \neq 0$, we would have that $\mathfrak{g}^\alpha$ acts locally nilpotently on M for some $\alpha \in \Delta^-_M$, which is impossible, as then $\operatorname{supp} M$ would have to be invariant with respect to the reflection $\sigma_\alpha \in W$. Since $\mathfrak{p}^{\mathrm{int}}_M \supset \mathfrak{h}$, $\mathfrak{p}^{\mathrm{int}}_M$ is a weight submodule of $\mathfrak{g}$ and $\mathfrak{p}^{\mathrm{int}}_M = \mathfrak{p}^{\mathrm{int}}_M \cap ((\mathfrak{g}^F_M + \mathfrak{g}^I_M) \oplus \mathfrak{g}^+_M) = (\mathfrak{g}^F_M + (\mathfrak{g}^I_M \cap \mathfrak{p}^{\mathrm{int}}_M)) \oplus \mathfrak{g}^+_M$.

(ii) The statement follows immediately from (i) since $\mathfrak{g}^I_M = \mathfrak{h} \subset \mathfrak{g}^F_M$ for a pseudo highest weight module M. ∎

The following proposition provides a more explicit description of $\mathfrak{p}_M = \mathfrak{p}^{\mathrm{int}}_M$ for highest weight modules. It is a version of the main result of [DP2] adapted to highest weight modules with respect to arbitrary Borel subalgebras of a root reductive Lie algebra $\mathfrak{g}$. If $M = V_{\mathfrak{b}}(\lambda)$, we say that a root $\alpha \in \Delta^-$ is M-*simple* if and only if $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Delta^-$ implies $\langle\lambda, \beta\rangle = 0$ or $\langle\lambda, \gamma\rangle = 0$.

Proposition 8. Let $M = V_{\mathfrak{b}}(\lambda)$ and let $\Sigma^{\mathrm{int}}_{\lambda,\mathfrak{b}}$ be the set of all M-simple roots $\delta \in \Delta^-$ for which M is $\mathfrak{g}^\delta$-integrable. Then, for any $\alpha \in \Delta^-$, M is $\mathfrak{g}^\alpha$-integrable (equivalently, $\alpha \in \Delta^- \cap \Delta^F_M$) if and only if $\alpha \in \langle\Sigma^{\mathrm{int}}_{\lambda,\mathfrak{b}}\rangle_{\mathbb{Z}_+}$. □

Proof. Proposition 7(ii) implies that if $\alpha \in \langle\Sigma^{\mathrm{int}}_{\lambda,\mathfrak{b}}\rangle_{\mathbb{Z}_+}$, then M is $\mathfrak{g}^\alpha$-integrable. Let, conversely, M be $\mathfrak{g}^\alpha$-integrable. If α is not M-simple, then $\alpha = \beta+\gamma$ for some $\beta, \gamma \in \Delta^-$ with $\langle\lambda, \beta\rangle \neq 0$ and $\langle\lambda, \gamma\rangle \neq 0$. The reader will then check that $|\langle\lambda, \alpha\rangle|$ is strictly bigger than both $|\langle\lambda, \beta\rangle|$ and $|\langle\lambda, \gamma\rangle|$. Choose n big enough so that $\alpha, \beta, \gamma \in \Delta_n$. Applying the main theorem from [DP2] to $V_{\mathfrak{b}_n}(\lambda|_{\mathfrak{h}_n})$, we obtain that $V_{\mathfrak{b}_n}(\lambda|_{\mathfrak{h}_n})$ is both $\mathfrak{g}^\beta$-integrable and $\mathfrak{g}^\gamma$-integrable. Therefore M is also $\mathfrak{g}^\beta$-integrable as well as $\mathfrak{g}^\gamma$-integrable. To conclude that $\alpha \in \langle\Sigma^{\mathrm{int}}_{\lambda,\mathfrak{b}}\rangle_{\mathbb{Z}_+}$ whenever $V_{\mathfrak{b}}(\lambda)$ is $\mathfrak{g}^\alpha$-integrable, one applies now induction on $|\langle\lambda, \alpha\rangle|$. ∎

Although the parabolic induction theorem does not hold for pseudo highest weight modules, their supports can be described explicitly. In the next section, we prove a general theorem describing the support of any irreducible weight module. An open question about pseudo highest weight modules is whether statement (ii) of Theorem 6 extends to any pseudo highest weight module; i.e., whether such a module is determined up to an isomorphism by its support.

6 The support of an arbitrary irreducible weight module

Let M be an arbitrary irreducible weight module M with corresponding partition $\Delta = \Delta^F_M \sqcup \Delta^I_M \sqcup \Delta^+_M \sqcup \Delta^-_M$. Define the *small Weyl group* W^F *of* M as the Weyl group of $\mathfrak{g}^F_M$. For $\lambda \in \mathfrak{h}^*$, set $K^\lambda_M := C^\lambda_M + \langle\Delta^I_M\rangle_{\mathbb{Z}} + \langle\Delta^-_M\rangle_{\mathbb{Z}_+}$ and $K^{\lambda,n}_M := C^{\lambda,n}_M + \langle\Delta^I_M \cap \Delta_n\rangle_{\mathbb{Z}} + \langle\Delta^-_M \cap \Delta_n\rangle_{\mathbb{Z}_+}$, where C^λ_M (resp., $C^{\lambda,n}_M$) is the intersection of the convex hull of $W^F \cdot \lambda$ (resp., of $(W^F \cap W_n) \cdot \lambda$) with $\lambda + \langle\Delta\rangle_{\mathbb{Z}}$.

Lemma 4. For any $\lambda \in \operatorname{supp} M$ and any n, there exists $\lambda_n \in \operatorname{supp} M$ such that $(\lambda+\langle\Delta_n\rangle_{\mathbb{R}}) \cap \operatorname{supp} M = K^{\lambda_n,n}_M$. □

Proof. If $\Delta_n \subset \Delta^I_M$, set $\lambda_n := \lambda$. Assume now that $\Delta_n \not\subset \Delta^I_M$. Consider the cone $K := \langle\Delta^F_M \cup \Delta^+_M\rangle_{\mathbb{R}_+}$. Then $(\lambda + K) \cap (\operatorname{supp} M \cap \langle\Delta_n\rangle_{\mathbb{R}})$ is a finite set. As in the proof of Theorem 2,

$\operatorname{supp} M = \cup_N \operatorname{supp} M_N$ for some irreducible $\mathfrak{g}_N$-modules M_N. Furthermore, there is N_0 for which $\operatorname{supp} M_{N_0} \supset (\lambda + K) \cap (\operatorname{supp} M \cap \langle \Delta_n \rangle_{\mathbb{R}})$. Let $\lambda_n \in \operatorname{supp} M_{N_0} \cap \langle \Delta_n \rangle_{\mathbb{R}}$ be such that $\lambda_n + \alpha \notin \operatorname{supp} M_{N_0}$ for any $\alpha \in \Delta_n \cap (\Delta_M^F \cup \Delta_M^+)$. (Such a weight λ_n exists because otherwise we would have $\Delta_n \subset \Delta_M^I$.) Clearly then, $(\lambda + \langle \Delta_n \rangle_{\mathbb{R}}) \cap \operatorname{supp} M = K_M^{\lambda_n, n}$. ■

Fix $\lambda \in \operatorname{supp} M$. Lemma 4 enables us to construct a sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that $(\lambda + \langle \Delta_n \rangle_{\mathbb{R}}) \cap \operatorname{supp} M = K_M^{\lambda_n, n}$. This sequence reconstructs $\operatorname{supp} M$ and is, in a certain sense, assigned naturally to M. For a precise formulation, define an equivalence relation $\sim_M$ on $\mathfrak{h}^*$ by setting $\lambda' \sim_M \lambda''$ if and only if there exists $w \in W^F$ such that $w(\lambda') - \lambda'' \in \langle \Delta_M^I \rangle_{\mathbb{Z}}$. Let furthermore $\mathfrak{h}_M^*$ denote the set of $\sim_M$-equivalence classes and let $p : \mathfrak{h}^* \mapsto \mathfrak{h}_M^*$ be the natural projection.

The following theorem is our most general result describing $\operatorname{supp} M$ for an arbitrary irreducible weight $\mathfrak{g}$-module M. It is a straightforward corollary of the construction of the sequence $\{\lambda_n\}$.

Theorem 8. (i) $K_M^{\lambda_1} \subset K_M^{\lambda_2} \subset \ldots$ and $\operatorname{supp} M = \cup_n K_M^{\lambda_n}$.

(ii) The sequence $\{p(\lambda_n)\}$ depends on λ only, and if λ' is another element of $\operatorname{supp} M$, there is n_0 so that $p(\lambda_n) = p(\lambda'_n)$ for all $n > n_0$.

(iii) If $\hat{M}$ is an irreducible weight $\mathfrak{g}$-module with corresponding sequence $\{\widehat{\lambda_n}\}$, then $\operatorname{supp} M = \operatorname{supp} \hat{M}$ if and only if there is n_0 such that $p(\lambda_n) = p(\widehat{\lambda_n})$ for $n > n_0$. □

The existence of a sequence of weights $\{\mu_n\}$ for which $\operatorname{supp} M = \cup_n K_M^{\mu_n}$ follows directly from the fact that $\operatorname{supp} M = \cup_n \operatorname{supp} M_n$ for some irreducible $\mathfrak{g}_n$-modules M_n (see the proof of Theorem 2). The main advantage of constructing the sequence $\{\lambda_n\}$ is that M stably determines the sequence $\{p(\lambda_n)\}$. It is an open question to describe explicitly all possible sequences $\{p(\lambda_n)\}$, or equivalently all possible supports of irreducible weight $\mathfrak{g}$-modules with a given shadow. We conclude this section with an example to Theorem 8.

Example 6. Let $M = V_{\mathfrak{b}}(\mu)$. Then $\mathfrak{g}_M^I = \mathfrak{h}$. Furthermore, for any fixed $\lambda \in \operatorname{supp} M$, there is N such that $\mu - \lambda \in \langle \Delta_N \rangle_{\mathbb{R}}$. If $\mathfrak{g}_M^F = \mathfrak{h}$, then Lemma 4 implies that $\lambda_n = \mu$ for any $n \geq N$, which explains the claim of Theorem 8 in this case. If $\mathfrak{g}_M^F \neq \mathfrak{h}$, then using Proposition 8, one checks that for $n \geq N$, λ_n can be any element of the orbit $(W^F \cap W_n) \cdot \mu$. This is in agreement with Theorem 8(ii) as, in this case, the projection p_M identifies all weights in any given orbit of W^F.

Appendix Borel subalgebras of $\mathfrak{g}$ and chains of subspaces in $\langle \Delta \rangle_{\mathbb{R}}$

In this Appendix, $\mathfrak{g}$ is an arbitrary Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ admits a root decomposition (1). Here we establish a precise interrelationship between

Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ (which we call simply Borel subalgebras), $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$, and oriented maximal chains of vector subspaces in $\langle\Delta\rangle_{\mathbb{R}}$. Our motivation is as follows. If $\mathfrak{g}$ has finitely many roots (i.e., for instance, if $\mathfrak{g}$ is finite-dimensional), then every Borel subalgebra is defined by a (nonunique) regular hyperplane H in $\langle\Delta\rangle_{\mathbb{R}}$, i.e., by a hyperplane H such that $H \cap \Delta = \emptyset$. If $\mathfrak{g}$ has infinitely many roots, this is known to be no longer true. That is, in general, there are Borel subalgebras which do not correspond to any regular hyperplane in $\langle\Delta\rangle_{\mathbb{R}}$. (This is the case for any Kac-Moody algebra which is not finite-dimensional.) In [DP1], we have shown that in the case where $\langle\Delta\rangle_{\mathbb{R}}$ is finite-dimensional, every Borel subalgebra can be defined by a maximal flag of vector subspaces in $\langle\Delta\rangle_{\mathbb{R}}$. However, when $\langle\Delta\rangle_{\mathbb{R}}$ is infinite-dimensional, the situation is more complicated and deserves a careful formulation.

We start by stating the relationship between Borel subalgebras $\mathfrak{b}$ with $\mathfrak{b} \supset \mathfrak{h}$ and $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$.

Proposition 9. Every $\mathbb{R}$-linear order on $\langle\Delta\rangle_{\mathbb{R}}$ determines a unique Borel subalgebra, and, conversely, every Borel subalgebra is determined by an (in general not unique) $\mathbb{R}$-linear order on $\langle\Delta\rangle_{\mathbb{R}}$. □

Proof. If $>$ is an $\mathbb{R}$-linear order on $\langle\Delta\rangle_{\mathbb{R}}$, then set $\Delta^{\pm} := \{\alpha \in \Delta \mid \pm\alpha > 0\}$. Then $\Delta^+ \sqcup \Delta^-$ is a triangular decomposition and thus $>$ determines the Borel subalgebra $\mathfrak{h} \oplus (\oplus_{\alpha\in\Delta^+}\mathfrak{g}^{\alpha})$. The proof of the converse statement is left to the reader (it is a standard application of Zorn's lemma). ■

In what follows, we establish a bijection between $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$ and oriented maximal chains of vector subspaces in $\langle\Delta\rangle_{\mathbb{R}}$. If V is any real vector space, a *chain of vector subspaces* of V is defined as a set of subspaces $F = \{F_\alpha\}_{\alpha\in A}$ of V such that $\alpha \neq \beta$ implies a proper inclusion $F_\alpha \subset F_\beta$ or $F_\beta \subset F_\alpha$. The set A is then automatically ordered. A chain F is a *flag* if and only if, as an ordered set, A can be identified with an (finite or infinite) interval in $\mathbb{Z}$. A chain F is *maximal* if and only if it is not properly contained in any other chain. Maximal chains of vector subspaces may be somewhat counterintuitive as the following example shows that a vector space of countable dimension admits noncountable maximal chains and vice versa.

Example 7. First, let V be a countable-dimensional vector space with a basis $\{e_r\}_{r\in\mathbb{Q}}$. The chain of all subspaces $V_t := \langle\{e_r \mid r < t\}\rangle_{\mathbb{R}}$ for $t \in \mathbb{R}$, $V'_q := \langle\{e_r \mid r \leq q\}\rangle_{\mathbb{R}}$ for $q \in \mathbb{Q}$, $V_{-\infty} := 0$, and $V_\infty := V$ is a maximal chain of cardinality continuum. Second, let $U := \mathbb{C}[[x]]$ be the space of formal power series in the indeterminate x. The dimension of U is continuum. However, if F denotes the flag $\{F_i := x^i U\}_{i\in\mathbb{Z}_+}$, then $\{0\} \cup F$ is a countable maximal chain in U.

Furthermore, it turns out that every maximal chain is determined uniquely by a certain subchain. A *maximal generalized flag*[2] is, by definition, a chain F which is minimal with the following property: for every $x \in V$, there exists a pair $F_\alpha \subset F_\beta \in F$ with $\dim F_\beta/F_\alpha = 1$ such that $x \in F_\beta \backslash F_\alpha$.

Lemma 5. Every maximal chain in V contains a unique maximal generalized flag and, conversely, every maximal generalized flag is contained in a unique maximal chain. □

Proof. Let $F = \{F_\alpha\}_{\alpha \in A}$ be a maximal chain in V. For every nonzero $x \in V$, set $F_x := \cup_{x \notin F_\alpha} F_\alpha$ and $F'_x := F_x \oplus \mathbb{R}x$. Then $F \cup \{F_x, F'_x\}$ is a chain in V and hence $F_x, F'_x \in F$. Let $G := \cup_{0 \neq x \in V}\{F_x, F'_x\}$. Then G is obviously a maximal generalized flag contained in F. Noting that G consists exactly of all pairs of subspaces from F with relative codimension one, we conclude that G is the unique maximal generalized flag contained in F.

Let now $G = \{G_\beta\}_{\beta \in B}$ be a maximal generalized flag in V and let $<$ be the corresponding order on B. A subset C of B is a *cut* of B if $\alpha \in C$ and $\beta < \alpha$ implies $\beta \in C$, and if $\alpha \in C$ and $\beta \notin C$ implies $\beta > \alpha$. The set $\bar{B}$ of all cuts of B is naturally ordered. Set $H_\alpha := \cup_{\beta \in \alpha} G_\beta$ for $\alpha \in \bar{B}$. One checks immediately that $H = \{H_\alpha\}_{\alpha \in \bar{B}}$ is a maximal chain in V which contains G. On the other hand, given any maximal chain $F = \{F_\alpha\}_{\alpha \in A}$ in V containing G, one notices that $F_\alpha = \cup_{G_\beta \subset F_\alpha} G_\beta$. Then the map $\varphi : A \to \bar{B}$, $\varphi(\alpha) := \{\beta \mid G_\beta \subset F_\alpha\}$ is an embedding of F into H and hence $F = H$. The lemma is proved. ■

An *orientation of a maximal generalized flag* G is a labeling of the two half-spaces of $G_\beta \backslash G_\alpha$, for every pair of indices $\alpha < \beta$ with $\dim G_\beta/G_\alpha = 1$, by mutually opposite signs $\pm$. An *orientation of a maximal chain* F is an orientation of the maximal generalized flag it contains. A maximal chain F is *oriented* if and only if an orientation of F is fixed.

Theorem 9. There is a bijection between oriented maximal chains in $\langle\Delta\rangle_{\mathbb{R}}$ and $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$. □

Proof. Lemma 5 implies that it is enough to establish a bijection between oriented maximal generalized flags in $\langle\Delta\rangle_{\mathbb{R}}$ and $\mathbb{R}$-linear orders on $\langle\Delta\rangle_{\mathbb{R}}$.

Let G be an oriented maximal generalized flag in $\langle\Delta\rangle_{\mathbb{R}}$. We define an $\mathbb{R}$-linear order $>_G$ on $\langle\Delta\rangle_{\mathbb{R}}$ by setting $x >_G 0$ or $x <_G 0$ according to the sign of the half-space of $G_\beta \backslash G_\alpha$ to which x belongs. Conversely, given an $\mathbb{R}$-linear order $>$ on $\langle\Delta\rangle_{\mathbb{R}}$, we build the corresponding oriented maximal generalized flag as follows. For a nonzero vector $x \in V$, we set $V_x := \{z \in V \mid \pm x > 0 \text{ implies } \pm(cz + x) > 0 \text{ for some } 0 \neq c \in \mathbb{R}_+\}$. Then for every

[2]Nonmaximal generalized flags can be defined, too. This is an interesting class of chains of subspaces which we intend to discuss elsewhere.

pair of nonzero vectors $x, y \in V$, exactly one of the following is true: $V_x = V_y$, $V_x \subset V_y$, or $V_x \supset V_y$. Therefore, the set of distinct subspaces among $\{V_x\}$ is a chain F. The fact that F is a maximal generalized flag in $\langle\Delta\rangle_{\mathbb{R}}$ is a direct consequence of its construction. To complete the proof of the theorem, it remains to orient F in the obvious way. ■

Finally we define a decomposition of Δ,

$$\Delta = \Delta^- \sqcup \Delta^0 \sqcup \Delta^+, \tag{8}$$

to be *parabolic* if and only if $\Pi(\Delta^-) \cap \Pi(\Delta^+) = \emptyset$, $0 \notin \Pi(\Delta^\pm)$, and $\Pi(\Delta)\backslash\{0\} = \Pi(\Delta^-) \sqcup \Pi(\Delta^+)$ is a triangular decomposition of $\Pi(\Delta)\backslash\{0\}$ (i.e., the cone $\langle\Pi(\Delta^+) \cup -\Pi(\Delta^-)\rangle_{\mathbb{R}_+}$ contains no vector subspace), where Π is the projection $\langle\Delta\rangle_{\mathbb{R}} \to \langle\Delta\rangle_{\mathbb{R}}/\langle\Delta^0\rangle_{\mathbb{R}}$. If $\Delta^0 = \emptyset$, a parabolic decomposition is a triangular decomposition. Given a parabolic decomposition (8), its corresponding *parabolic subalgebra* is by definition $\mathfrak{p} := \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^0 \sqcup \Delta^+} \mathfrak{g}^\alpha)$.

Given a linear subspace V' in V, we define a V'-*maximal* chain of vector subspaces in V to be the preimage in V of a maximal chain in V/V'. A V'-maximal chain is *oriented* if and only if the corresponding maximal chain in V/V' is oriented. Generalizing the corresponding construction for Borel subalgebras, we have the following theorem.

Theorem 10. Any parabolic decomposition (8) is determined by some oriented $\langle\Delta^0\rangle_{\mathbb{R}}$-maximal chain of vector subspaces in V, and, conversely, any oriented V'-maximal chain in $\langle\Delta\rangle_{\mathbb{R}}$, for an arbitrary subspace V' of $\langle\Delta\rangle_{\mathbb{R}}$, defines a unique parabolic decomposition of Δ with $\Delta^0 = \Delta \cap V'$. □

Proof. The proof is an exercise. ■

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