A BOTT-BOREL-WEIL THEORY FOR DIRECT LIMITS OF ALGEBRAIC GROUPS

By Ivan Dimitrov, Ivan Penkov and Joseph A. Wolf

Abstract. We develop a Bott-Borel-Weil theory for direct limits of algebraic groups. Some of our results apply to locally reductive ind-groups G in general, i.e., to arbitrary direct limits of connected reductive linear algebraic groups. Our most explicit results concern root-reductive ind-groups G, the locally reductive ind-groups whose Lie algebras admit root decomposition. Given a parabolic subgroup P of G and a rational irreducible P-module, we consider the irreducible G-sheaves $\mathcal{O}_{G/P}(E)$ and their duals $\mathcal{O}_{G/P}(E^*)$. These sheaves are locally free, in general of infinite rank. We prove a general analog of the Bott-Borel-Weil Theorem for $\mathcal{O}_{G/P}(E^*)$, namely that $H^q(G/P; \mathcal{O}_{G/P}(E^*))$ is nonzero for at most one index $q = q_0$ and that $H^{q_0}(G/P; \mathcal{O}_{G/P}(E^*))$ is isomorphic to the dual of a rational irreducible G-module V. For $q_0 > 0$ we show that (in contrast to the finite dimensional case) V need not admit an irreducible P-submodule. There, however, one has a larger parabolic subgroup ${}^wP \supset P$, constructed from P and a Weyl group element w of length q_0, such that V is generated by an irreducible ${}^wP$-submodule. Consequently certain G-modules V can appear only for $q_0 > 0$, never for $q_0 = 0$. For $\mathcal{O}_{G/P}(E)$ we show that there is no analog of Bott's vanishing theorem, more precisely that $\mathcal{O}_{G/P}(E)$ can have arbitrarily many nonzero cohomology groups. Finally, we give an explicit criterion for the projectivity of the ind-variety G/P, showing that G/P is in general not projective.

0. Introduction. The classical Bott-Borel-Weil Theorem [5] is the cornerstone of the geometric approach to representation theory. Analogs of this theorem have been studied in various contexts, including that of homogeneous spaces in characteristic p [14] and of homogeneous superspaces [23]. In these contexts the Bott-Borel-Weil Theorem does not carry over as a single theorem, and this has inspired important areas of investigation. Infinite dimensional group analogs of the Bott-Borel-Weil Theorem have also been studied. The loop group case was addressed in the 1980's; see [18] and [24]. Direct limit Lie groups were first addressed recently by L. Natarajan, E. Rodríguez-Carrington and one of us [20]. Roughly speaking, the results of [20] extend the finite dimensional analytic Bott-Borel-Weil Theorem to direct limit Lie groups and direct limit unitary representations, in other words to the analytic category with representations on Hilbert spaces.

In this paper we view the classical Bott-Borel-Weil Theorem as a result in algebraic geometry, see [6] and [7] (B. Kostant's purely algebraic version of this

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theorem, [16], is beyond our scope here), and try to find its analog for direct limits of linear algebraic groups. This turns out to be a very interesting problem and, as in the characteristic p and the supergeometry contexts, one is led not to a single theorem but to the development of a new circle of ideas. To start, recall two forms of the Bott-Borel-Weil Theorem for a connected complex reductive linear algebraic group G:

(1) Let B be a Borel subgroup of G, L an irreducible (one dimensional) rational B-module, $\mathbb{L} \to G/B$ the associated homogeneous line bundle, and $\mathcal{O}_{G/B}(L)$ be the sheaf of regular local sections of $\mathbb{L}$. Then at most one of the cohomology groups $H^q(G/B; \mathcal{O}_{G/B}(L))$ is nonzero, and if $H^{q_0}(G/B; \mathcal{O}_{G/B}(L)) \neq 0$ then $H^{q_0}(G/B; \mathcal{O}_{G/B}(L))$ is an irreducible G-module whose highest weight has an explicit expression in terms of the weight of L.

(2) Let P be a parabolic subgroup of G with $B \subset P$ and let π: $G/B \to G/P$ denote the canonical projection. Let E be an irreducible (finite dimensional) rational P-module, $\mathbb{E} \to G/P$ be the associated homogeneous vector bundle, and $\mathcal{O}_{G/P}(E)$ be the sheaf of regular local sections of $\mathbb{E}$. Then $\mathcal{O}_{G/P}(E)$ is isomorphic to the direct image sheaf $\pi_*\mathcal{O}_{G/B}(L)$ for some one dimensional B-module L, and $H^q(G/P; \mathcal{O}_{G/P}(E)) = H^q(G/B; \mathcal{O}_{G/B}(L))$ for any q. Therefore at most one cohomology group $H^{q_0}(G/P; \mathcal{O}_{G/P}(E))$ is nonzero, and it is an irreducible G-module whose highest weight can be calculated explicitly.

Statement (1) admits a reasonably straightforward generalization to locally reductive ind-groups. As follows from Theorem 11.1(ii) below, if G is a locally reductive ind-group, B is a Borel subgroup, and L is any irreducible rational (and thus one dimensional) B-module, then the sheaf $\mathcal{O}_{G/B}(L)$ has at most one nonzero cohomology group $H^{q_0}(G/B; \mathcal{O}_{G/B}(L))$. Furthermore, $H^{q_0}(G/B; \mathcal{O}_{G/B}(L))$ is isomorphic to the dual of an irreducible rational G-module V, and therefore (in contrast to the classical case and to the analytic category case [20]) is a reducible G-module except in the very special situation of finite dimensional V. If $q_0 = 0$, V has a highest weight which is the negative of the weight of L. For $q_0 > 0$ the explicit description of V in terms of L is an interesting problem, and it is still open for sufficiently general ind-groups G. However, for a root-reductive ind-group G (root-reductive ind-groups are the simplest ind-versions of reductive groups, see Section 4) we prove in Proposition 14.1 that V is a highest weight module whose highest weight is calculated in terms of L using a Weyl group action which has many properties in common with that of the classical Bott-Borel-Weil Theorem.

In this paper we develop a Bott-Borel-Weil theory for locally reductive ind-groups G, which in particular replaces (2). It involves a number of new constructions, and goes well beyond straightforward generalization. To indicate why this is needed, here are two observations on a parabolic subgroup P of G and an infinite dimensional irreducible rational P-module E. First, E need not be locally irreducible, i.e., need not be isomorphic to a direct limit of irreducible P_n-modules, and also E need not be a weight module. In particular, E need not have an extremal weight with respect to any Borel subgroup $B \subset P$. Second, there are two natural G-sheaves to consider: the sheaf $\mathcal{O}_{G/P}(E)$ of local sections of the G-bundle $\mathbb{E} \to G/P$ associated to E, and the sheaf $\mathcal{O}_{G/P}(E^*)$ of local sections of the dual bundle (of uncountable rank) $\mathbb{E}^* \to G/P$. It is not difficult to see that the "push down approach" of (2) applies only to $\mathcal{O}_{G/P}(E^*)$ and then only under the additional condition that E be a highest weight P-module. Thus the "push down approach" does not lead to a complete description of the cohomology of $\mathcal{O}_{G/P}(E)$ or $\mathcal{O}_{G/P}(E^*)$.

Our main results, collected in Theorem 11.1, concern the sheaf $\mathcal{O}_{G/P}(E^*)$. When G is root-reductive, but E need not be locally irreducible, we prove that $\mathcal{O}_{G/P}(E^*)$ has at most one nonzero cohomology group $H^{q_0}(G/P; \mathcal{O}_{G/P}(E^*))$, and that this cohomology group is isomorphic to the dual of a rational G-module V. For $q_0 = 0$, we show that, as in the finite dimensional case, V is parabolically generated by E itself. When E is a weight module this latter result turns out to imply that $H^0(G/P; \mathcal{O}_{G/P}(E^*)) \neq 0$ forces E to be locally irreducible; more precisely, E has to be a finite P-module (see Theorem 11.1(iv)). For $q_0 > 0$, in contrast with the finite dimensional case, V need not be generated by an irreducible P-submodule. We prove that here V is generated by an irreducible submodule of a larger parabolic subgroup ${}^wP \supset P$ that depends on P and a certain element w of the Weyl group. The result is sharp in the sense that there are examples of irreducible P-modules E for which V does not admit an irreducible submodule for any parabolic subgroup of G properly contained in ${}^wP$; see Example 13.1. This new infinite dimensional phenomenon is quite remarkable, for it provides a geometric construction of G-modules that are not parabolically generated, in particular of cuspidal weight modules [8]. According to Theorem 11.1(iv), cuspidal weight G-modules can only occur in higher cohomology groups. As a consequence, there is no "Demazure isomorphism" between the unique nonzero higher cohomology group of $\mathcal{O}_{G/P}(E^*)$ and the zeroth cohomology group of $\mathcal{O}_{G/P}(E'^*)$ for any other E'.

In connection with the above results, we establish an explicit criterion for the projectivity of the ind-variety G/P. Somewhat surprisingly, it turns out that G/P is rarely projective, even for $G = GL(\infty)$; see §15. Finally, we consider the sheaf $\mathcal{O}_{G/P}(E)$ and show by an example that it can have arbitrarily many nonzero cohomology groups. The problem of a systematic description of the cohomology for sheaves of type $\mathcal{O}_{G/P}(E)$ remains open.

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Notational conventions. The ground field is $\mathbb{C}$, though all results extend easily to any algebraically closed field of characteristic zero. If V is a vector space, V^* stands for its dual space. $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, $\mathbb{Z}_- = -\mathbb{Z}_+$, $\mathbb{N} := \mathbb{Z}_+ \backslash \{0\}$, and $\mathbb{C}^\times$ is the multiplicative group $\{a \in \mathbb{C} \mid a \neq 0\}$. Expressions such as $\varinjlim G_n$ and $\varprojlim H^q(X_n; \ldots)$ denote the direct and inverse limit, respectively of a direct or inverse system indexed by $\mathbb{Z}_+$. The notation $H^q(X; \mathcal{F})$ always stands for the q^{th} cohomology group of a sheaf $\mathcal{F}$ on a topological space X. If $\pi\colon X \to Y$ is a morphism of algebraic varieties, π^* and π_* denote respectively the inverse and direct image functors of coherent sheaves. Ind-groups (in particular complex algebraic groups) are denoted by capital letters (e.g. K), and their Lie algebras are denoted by the corresponding lower case Gothic letter (e.g. $\mathfrak{k}$). Furthermore, as the ground field is fixed, we will write simply $GL(n)$, $SO(n)$, etc. instead of $GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, etc. $\mathcal{U}(\mathfrak{k})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{k}$. The sign $\niplus$ stands for semidirect sum of Lie algebras: if $\mathfrak{k} = \mathfrak{m} \niplus \mathfrak{n}$, then $\mathfrak{m}$ is an ideal in $\mathfrak{k}$ and $\mathfrak{n}$ is a complementary subalgebra. The sign $\overset{\mathrm{f}}{\times}$ denotes "finite" or "weak" direct product of groups or of homogeneous spaces. If G^t, $t \in T$, is an infinite family of groups, we set

$$\overset{\mathrm{f}}{\times}_{t \in T} G^t := \{\times_{t \in T}\, g^t \mid g^t \in G^t \text{ for all } t \text{ and } g^t = 1 \text{ for almost all } t \in T\}.$$

Similarly, if G^t/H^t is a family of homogeneous spaces, $\overset{\mathrm{f}}{\times}_{t \in T} (G^t/H^t)$ is the image of $\overset{\mathrm{f}}{\times}_{t \in T} G^t$ in $\times_{t \in T} (G^t/H^t)$. Finally, if $\mathfrak{k}' \subset \mathfrak{k}$ as a pair of finite dimensional Lie algebras, and $E_{\mathfrak{k}}$ and $E_{\mathfrak{k}'}$ are respectively a finite dimensional irreducible $\mathfrak{k}$-module and a finite dimensional irreducible $\mathfrak{k}'$-module, we write $E_{\mathfrak{k}'} \prec E_{\mathfrak{k}}$ if there is an injection of $\mathfrak{k}'$-modules $E_{\mathfrak{k}'} \to E_{\mathfrak{k}}$.

Part I. Ind-varieties and Ind-groups.

1. Ind-varieties. This is a quick summary of the basic definitions on ind-varieties. See both [25] and [17] for more detailed summaries.

An *ind-variety* (over $\mathbb{C}$) is a set X with a filtration

$$X_0 \subset X_1 \subset X_2 \subset \cdots \tag{1.1}$$

such that $X = \bigcup_{n \geq 0} X_n$, each X_n is a Noetherian algebraic variety, and the inclusions $X_n \subset X_{n+1}$ are closed immersions of algebraic varieties. In the following we will often write $X = \varinjlim X_n$. An ind-variety X is automatically a topological space: a subset $U \subset X$ is *open* in X if and only if, for each n, $U \cap X_n$ is an open subvariety of X_n. The *sheaf of regular functions on* X, or *the structure sheaf* $\mathcal{O}_X$ of X, is the inverse limit $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$ of the sheaves of regular functions $\mathcal{O}_{X_n}$ on the X_n. An ind-variety $X = \varinjlim X_n$ is *proper* if and only if all the varieties X_n are proper, is *affine* if and only if all the X_n are affine. A *morphism* from an ind-variety X to an ind-variety Y is a map $\varphi\colon X \to Y$ such that, for every

$n \geq 0$, the restriction $\varphi|_{X_n}$ is a morphism of X_n into Y_m for some $m = m(n)$. An *isomorphism* of ind-varieties is a morphism which admits an inverse morphism. An *ind-subvariety* Z of X is a subset $Z \subset X$ such that $Z \cap X_n$ is a subvariety of X_n for each n. An ind-variety is *connected* if it is connected as a topological space.

The (*Zariski*) *tangent space* $T_x(X)$, to an ind-variety $X = \varinjlim X_n$ at a closed point $x \in X$ is, by definition, the direct limit $\varinjlim T_x(X_n)$ where $x \in X_n$ for n sufficiently large. Any ind-variety morphism φ: $X \to Y$ induces linear maps $d\phi_x$: $T_x(X) \to T_{\phi(x)}(Y)$ for all closed points $x \in X$.

Example 1.2.

(1) $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n = \{(a_1, a_2, a_3, \ldots) \mid a_n \in \mathbb{C}$, only finitely many $a_n \neq 0\}$ is an affine ind-variety.

(2) Every (complex) vector space V of countable dimension has a canonical structure of an affine ind-variety: any basis $\{v_1, \ldots, v_n, \ldots\}$ identifies $V = \varinjlim \mathrm{Span}\{v_1, \ldots, v_n\}$ with $\mathbb{C}^\infty$ as sets and thus defines an ind-variety structure on V. Any other basis of V defines the same ind-variety structure because the identity map on V is an ind-variety isomorphism.

(3) Let V be a vector space of countable dimension. Fix an integer $k \geq 1$. The set $Gr(k, V)$ of all k-dimensional subspaces of V has a canonical structure of proper ind-variety: any filtration $0 \subset V_k \subset V_{k+1} \subset \cdots \subset V = \bigcup_{r \geq 0} V_{k+r}$, $\dim V_{k+r} = k + r$, induces a filtration

$$Gr(k, V_k) \subset Gr(k, V_{k+1}) \subset \cdots \subset Gr(k, V),$$

and the associated ind-variety structure on $Gr(k, V)$ is independent of choice of filtration on V. For $k = 1$, $\mathbb{P}(V) := Gr(1, V)$ is by definition the *projective ind-space associated to* V.

2. Projective and locally projective ind-varieties. An ind-variety X is *locally projective* if it admits an ind-variety filtration (1.1) such that all the X_n are projective varieties. An ind-variety X is *projective* if it can be embedded as a closed ind-subvariety into the projective ind-space $\mathbb{P}(\mathbb{C}^\infty)$. (Tjurin's notion of projectivity [26] is stronger, as it requires the existence of a finite codimensional projective embedding.) Any projective ind-variety is proper and locally projective, but the converse is not true. Below we introduce twisted projective ind-spaces which are the simplest examples of locally projective, generically not projective, ind-varieties.

Let Y be a proper Noetherian algebraic variety. A very ample invertible $\mathcal{O}_Y$-module $\mathcal{L}_Y$ (i.e., a locally free sheaf $\mathcal{L}_Y$ of $\mathcal{O}_Y$-modules of rank 1 which is generated by its global sections) determines a closed immersion of Y into the projective space $\mathbb{P} := \mathbb{P}(H^0(Y; \mathcal{L}_Y)^*)$, and $\mathcal{L}_Y$ is identified with the inverse image under this immersion of the standard sheaf $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}$. See [12, II, §5]. If X is a proper ind-variety, consider an invertible $\mathcal{O}_X$-module $\mathcal{L}_X$ and an ind-variety filtration (1.1) such that all inverse images $\mathcal{L}_{X_n}$ of $\mathcal{L}_X$ are very ample and the restriction maps $H^0(X_n;\mathcal{L}_{X_n}) \to H^0(X_{n-1};\mathcal{L}_{X_{n-1}})$ are surjective. Then the system dual to the inverse system

$$\cdots \to H^0(X_n;\mathcal{L}_{X_n}) \to H^0(X_{n-1};\mathcal{L}_{X_{n-1}}) \to \cdots \to H^0(X_0;\mathcal{L}_{X_0}) \to 0$$

is a direct system of injections, and $\mathcal{L}_X$ defines a closed immersion of X into $\mathbb{P}(\varinjlim H^0(X_n;\mathcal{L}_{X_n})^*)$.

We define X to be a *twisted projective ind-space* if it admits a filtration (1.1) such that X_n is a projective space for all n. To a twisted projective ind-space we attach its *twisting sequence* $\{c_1, c_2, \ldots\}$, where c_n is the natural number that denotes the first Chern class of the inverse image of $\mathcal{O}_{X_{n+1}}(1)$ on X_n. Any sequence of natural numbers can be obtained as a twisting sequence.

PROPOSITION 2.1. *A twisted projective ind-space X is projective if and only if its twisting sequence stably equals the sequence* $\{1, 1, \ldots\}$, *i.e.* $c_n = 1$ *for n sufficiently large.*

Proof. If $c_n = 1$ for all $n > n_0$, then $X_{n_0} \subset X_{n_0+1} \subset \cdots$ is an ind-variety filtration for X and, for any $n \geq n_0$, the invertible $\mathcal{O}_{X_n}$-module $\mathcal{O}_{X_n}(1)$ with Chern class 1 is the inverse image of a well-defined $\mathcal{O}_X$-module $\mathcal{O}_X(1)$. Then $\mathcal{O}_X(1)$ establishes an isomorphism between X and the projective ind-space $\mathbb{P}(\varinjlim H^0(X_n;\mathcal{O}_{X_n}(1))^*)$. Thus X itself is isomorphic to a projective ind-space. Conversely, let the sequence $c_1, c_2, \ldots$ corresponding to X admit a subsequence $c_{n_1}, c_{n_2}, \ldots$ with $c_{n_t} \geq 2$ for all t. Assume that X is a closed ind-subvariety of a projective ind-space $\mathbb{P}$ and consider the inverse images $\mathcal{O}_{\mathbb{P}}(1)_{|}X_n$ of $\mathcal{O}_{\mathbb{P}}(1)$ on X_n, where $\mathcal{O}_{\mathbb{P}}(1) := \varprojlim \mathcal{O}_{\mathbb{P}_n}(1)$. Denote the Chern class of $\mathcal{O}_{\mathbb{P}}(1)_{|}X_n$ by C_n. Then, for any $s > k$, $C_k/C_s = c_{k+1}c_{k+2}\cdots c_s$. Therefore the positive integer C_{n_0} is divisible by all products $c_{n_0+1}c_{n_0+2}\cdots c_n$ for any $n > n_0$, which is an obvious contradiction. $\square$

We remark that the ind-grassmannian $Gr(k, V)$ of Example 1.2(3) is projective, and that the classical Plücker embeddings induce a closed immersion $Pl^k\colon\ Gr(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$ where $\bigwedge^k$ denotes k^{th} exterior power.

3. Ind-groups and direct limit Lie algebras. An *algebraic ind-group*, or, briefly, *ind-group*, is an ind-variety G with group structure such that the map

$$G \times G \to G, \qquad (g_1, g_2) \mapsto g_1 g_2^{-1}$$

is a morphism of ind-varieties. By definition, an *ind-group homomorphism* is a group homomorphism $\kappa\colon\ G \to K$ that is also an ind-variety morphism. An *ind-subgroup* K of G is a subgroup $K \subset G$ that is an ind-subvariety.

We define a *locally linear ind-group* as an (affine) ind-variety G with an ind-variety filtration

$$G_0 \subset G_1 \subset G_2 \subset \cdots \tag{3.1}$$

such that all the G_n are linear algebraic groups and all the inclusions are closed immersions that are group morphisms. Clearly, every locally linear ind-group is an affine ind-group but the converse is not true. An example of an affine ind-group which is not locally linear is provided by the automorphism group $\mathrm{Aut}(\mathbb{A}^m)$ of the m-dimensional affine space $\mathbb{A}^m$; see [25]. We will only study connected locally linear ind-groups, and all ind-groups considered below are assumed to be connected and locally linear. By a slight abuse of language we refer to them simply as ind-groups.

Let G be a (connected) ind-group. In this paper we define a *parabolic subgroup* of G as an ind-subgroup P of G such that, for a suitable filtration (3.1), $P_n := P \cap G_n$ is a parabolic subgroup of G_n for each n, and, in addition, $U_{P_{n-1}} = U_{P_n} \cap P_{n-1}$, where U_{P_i} denotes the unipotent radical of P_i. Similarly, a *Borel subgroup* of G is an ind-subgroup B of G such that, for a suitable filtration (3.1), $B_n := B \cap G_n$ is a Borel subgroup of G_n for each n (the condition on the unipotent radicals is automatic here), and a *Cartan subgroup* of G is an ind-subgroup H of G such that, for a suitable filtration (3.1), $H_n := H \cap G_n$ is a Cartan subgroup of G_n for each n. An element $u \in G$ is *unipotent* if for some (or equivalently, for any) ind-group filtration (3.1) u is unipotent in G_n whenever n is large enough so that $u \in G_n$. The *unipotent radical* U_G of G is defined to be the largest closed normal ind-subgroup of G such that every element u is unipotent. Finally, an ind-group G is *locally reductive* if we can choose the ind-group filtration (3.1) so that each G_n is a reductive linear algebraic group. Whenever $G = \varinjlim G_n$ is locally reductive we will assume that the G_n are also reductive.

Throughout the rest of the paper we will consider ind-groups G with a fixed filtration (3.1) of connected linear algebraic groups. We will assume without explicit mention that the parabolic, Borel and Cartan subgroups of G are aligned with respect to that filtration in the above sense.

PROPOSITION 3.2. *Let $G = \varinjlim G_n$ be a locally reductive ind-group and let $P = \varinjlim P_n$ be a parabolic ind-subgroup.*

(i) *The unipotent radical U_P of P is well-defined, and $U_P = \varinjlim U_{P_n}$.*

(ii) *The Chevalley semidirect product decompositions $P_n = U_{P_n} \rtimes P_n^{\mathrm{red}}$, into the unipotent radical and a complementary reductive subgroup, can be chosen so that $P_n^{\mathrm{red}} \subset P_{n+1}^{\mathrm{red}}$ for all n. Then $P = U_P \rtimes P^{\mathrm{red}}$ where $P^{\mathrm{red}} := \varinjlim P_n^{\mathrm{red}}$.*

Proof. According to the definition, $\varinjlim U_{P_n}$ is a well-defined ind-subgroup of G, which is closed and normal in P since, for each n, U_{P_n} is closed and normal in P_n. Furthermore, $\varinjlim U_{P_n}$ is the largest closed normal subgroup of P in which every element is unipotent, because the existence of a larger such subgroup would contradict the fact that U_{P_n} is the unipotent radical of P_n for every n. Therefore $\varinjlim U_{P_n}$ is the unipotent radical U_P of P.

A theorem of Mostow [19, Theorem 7.1] ensures that the maximal reductive subgroups of P_{n+1} are just the reductive subgroups R such that $P_{n+1} = U_{P_{n+1}} \rtimes R$, and that any two such groups R are conjugate by an element of $U_{P_{n+1}}$. See [13, VIII, Theorem 4.3] for a systematic development. Therefore, given a maximal reductive subgroup P_n^{red} in P_n, we can choose P_{n+1}^{red} to be any maximal reductive subgroup of P_{n+1} that contains it. This implies (ii), and the equality $P = U_P \rtimes P^{\mathrm{red}}$ follows. □

The *Lie algebra* of an ind-group $G = \varinjlim G_n$ is the direct limit Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ for the direct system

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots, \tag{3.3}$$

where $\mathfrak{g}_n$ is the Lie algebra of G_n and the inclusions $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ are the differentials of the group immersions $G_n \subset G_{n+1}$. This Lie algebra is the tangent space $T_{1_G}(G)$ with its natural Lie algebra structure [17]. An ind-group homomorphism κ: $G \to K$ induces a Lie algebra homomorphism $d\kappa$: $\mathfrak{g} \to \mathfrak{k}$ [17]. We shall consider direct limit Lie algebras more generally, sometimes without regard to ind-groups. These will always correspond to direct systems of injections of finite dimensional Lie algebras.

Let $\mathfrak{g}$ be a direct limit Lie algebra. In this paper we define a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ to be a *parabolic subalgebra* if, for a suitable direct system $\{\mathfrak{g}_n\}$, $\mathfrak{p}_n := \mathfrak{p} \cap \mathfrak{g}_n$ is a parabolic subalgebra of $\mathfrak{g}_n$ for each n, and, in addition, $\mathfrak{u}_{\mathfrak{p}_{n-1}} = \mathfrak{u}_{\mathfrak{p}_n} \cap \mathfrak{p}_{n-1}$, where $\mathfrak{u}_{\mathfrak{p}_i}$ denotes the nilpotent radical of $\mathfrak{p}_i$. Similarly, a subalgebra $\mathfrak{b} \subset \mathfrak{g}$ (respectively $\mathfrak{h} \subset \mathfrak{g}$) is a *Borel* (respectively *Cartan*) *subalgebra* of $\mathfrak{g}$ if, for a suitable direct system $\{\mathfrak{g}_n\}$, each $\mathfrak{b}_n := \mathfrak{b} \cap \mathfrak{g}_n$ (respectively $\mathfrak{h}_n := \mathfrak{h} \cap \mathfrak{g}_n$) is a Borel subalgebra (respectively Cartan subalgebra) of $\mathfrak{g}_n$. (This definition of parabolic and Borel subalgebras is more general than that of [8].) In the rest of the paper we will automatically assume that $\mathfrak{g}$ is equipped with a fixed filtration (3.3) and that all parabolic, Borel or Cartan subalgebras we consider are aligned with respect to that filtration in the above sense.

We define a direct limit Lie algebra $\mathfrak{g}$ to be *locally reductive* if we have an expression $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ where each of the $\mathfrak{g}_n$ is reductive. Whenever we express a locally reductive Lie algebra $\mathfrak{g}$ as $\varinjlim \mathfrak{g}_n$, it will be assumed that the $\mathfrak{g}_n$ are reductive. The following Proposition is the Lie algebra analog of Proposition 3.2. Part (ii) of this Proposition is an adaptation of a result of Baranov [2, Lemma 4.3]. We leave the proof to the reader.

PROPOSITION 3.4. *Let $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ be a locally reductive direct limit Lie algebra and let $\mathfrak{p}$ be a parabolic subalgebra.*

(i) $\mathfrak{u}_{\mathfrak{p}} := \varinjlim \mathfrak{u}_{\mathfrak{p}_n}$ *is a well-defined ideal in $\mathfrak{p}$. By definition, $\mathfrak{u}_{\mathfrak{p}}$ is the nilpotent radical of $\mathfrak{p}$.*

(ii) *One can choose semidirect sum decompositions* $\mathfrak{p}_n = \mathfrak{u}_{\mathfrak{p}_n} \inplus \mathfrak{p}_n^{\mathrm{red}}$, *into the nilpotent radical of* $\mathfrak{p}_n$ *and a complementary reductive subalgebra, such that each* $\mathfrak{p}_n^{\mathrm{red}} \hookrightarrow \mathfrak{p}_{n+1}^{\mathrm{red}}$. *Then* $\mathfrak{p} = \mathfrak{u}_{\mathfrak{p}} \inplus \mathfrak{p}^{\mathrm{red}}$ *where* $\mathfrak{p}^{\mathrm{red}} := \varinjlim \mathfrak{p}_n^{\mathrm{red}}$.

(iii) *Let* $\mathfrak{g}$ *be the Lie algebra of a reductive ind-group G, P is a parabolic ind-subgroup of G with Lie algebra* $\mathfrak{p}$, *and U_P be the unipotent radical of P. Then* $\mathfrak{u}_{\mathfrak{p}}$ *is the Lie algebra of U_P. Furthermore, if we choose* $\mathfrak{p}_n^{\mathrm{red}}$ *to be the Lie algebra of* P_n^{red} *of Proposition 3.2 (ii), then* $\mathfrak{p}^{\mathrm{red}}$ *is the Lie algebra of* P^{red}.

4. Root-reductive ind-groups and parabolic subgroups. Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be Lie algebras with root decomposition, so $\mathfrak{g}' = \mathfrak{h}' \oplus (\bigoplus_{\alpha' \in \Delta'} (\mathfrak{g}')^{\alpha'})$ and $\mathfrak{g}'' = \mathfrak{h}'' \oplus (\bigoplus_{\alpha'' \in \Delta''} (\mathfrak{g}'')^{\alpha''})$. Here $\mathfrak{h}'$ and $\mathfrak{h}''$ are respectively Cartan subalgebras, and Δ' and Δ'' are the corresponding root systems of $\mathfrak{g}'$ and $\mathfrak{g}''$. A Lie algebra homomorphism $\varphi\colon \mathfrak{g}' \to \mathfrak{g}''$ is a *root homomorphism*, if $\varphi(\mathfrak{h}') \subset \mathfrak{h}''$ and φ maps every root space $(\mathfrak{g}')^{\alpha'}$ into a root space $(\mathfrak{g}'')^{\alpha''}$, thus mapping Δ' into Δ''. To be precise one should write $\varphi\colon (\mathfrak{g}', \mathfrak{h}', \Delta') \to (\mathfrak{g}'', \mathfrak{h}'', \Delta'')$, but we often leave this to be understood by the reader. A *root subalgebra* of $\mathfrak{g}''$ is the image of a root homomorphism. A locally reductive direct limit Lie algebra $\mathfrak{g}$ is *root-reductive* if it can be expressed as a direct limit $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ where all $\mathfrak{g}_n$ are finite dimensional and reductive, $\mathfrak{h} = \varinjlim \mathfrak{h}_n$ is a Cartan subalgebra, and each injection $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ carries $\mathfrak{h}_n$ into $\mathfrak{h}_{n+1}$ and is a root homomorphism. Finally, a locally reductive ind-group G is *root-reductive* if its Lie algebra $\mathfrak{g}$ is root-reductive.

Let $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ be a root-reductive direct limit Lie algebra, expressed as a direct limit of root injections $(\mathfrak{g}_n, \mathfrak{h}_n, \Delta_n) \hookrightarrow (\mathfrak{g}_{n+1}, \mathfrak{h}_{n+1}, \Delta_{n+1})$. Then $\mathfrak{g}$ admits a root decomposition with respect to the Cartan subalgebra $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$. The root system of $(\mathfrak{g}, \mathfrak{h})$ is $\Delta = \bigcup_n \Delta_n$, and all root spaces $\mathfrak{g}^\alpha$ are one dimensional. It is easy to check (see [8]) that the direct limit $\mathcal{W} = \varinjlim \mathcal{W}_n$ of the Weyl groups $\mathcal{W}_n$ (considered as the groups generated by root reflections) of $\mathfrak{g}_n$ is well defined. In this paper $\mathcal{W}$ is by definition the *Weyl group of* $\mathfrak{g}$. Let furthermore $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra such that $\mathfrak{h} \subset \mathfrak{b}$. It determines a decomposition $\Delta = \Delta^+ \sqcup \Delta^-$ such that $\Delta^- = -\Delta^+$, the positive roots Δ^+ being the roots of $\mathfrak{b}$. We say that a positive root α is $\mathfrak{b}$-*simple* if α cannot be decomposed as the sum of two positive roots. A Weyl group element $w \in \mathcal{W}$ is of *finite length with respect to* $\mathfrak{b}$ if w is a (finite) product of simple root reflections, $w = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_k}$ for some $\mathfrak{b}$-simple roots $\alpha_1, \dots, \alpha_k$. The length of the shortest such expression is by definition the *length of w with respect to* $\mathfrak{b}$. We remark also that a subalgebra $\mathfrak{p}$ of a root-reductive direct limit Lie algebra $\mathfrak{g}$ is parabolic if and only if $\mathfrak{p}_n$ is a parabolic subalgebra of $\mathfrak{g}_n$ for each n, as in this case the condition on the nilpotent radicals is automatically satisfied.

Root-reductive direct limit Lie algebras were introduced and studied in [8]; also see [20, Section 7]. They are closely related to the *classical simple direct limit Lie algebras* $\mathfrak{a}(\infty)$, $\mathfrak{b}(\infty)$, $\mathfrak{c}(\infty)$ and $\mathfrak{d}(\infty)$ defined by letting $\mathfrak{g}_n$ be the corresponding finite dimensional Lie algebra of rank n and by requiring that all φ_n be root injections. The isomorphism class of the resulting direct limit Lie algebra does not depend on the injections φ_n [8]. Moreover, every simple infinite dimensional root-reductive Lie algebra is isomorphic to one of the four classical simple direct limit Lie algebras. (A further interesting fact is that $\mathfrak{b}(\infty)$ and $\mathfrak{d}(\infty)$ are isomorphic as Lie algebras, see [3] and [22]. However, no such isomorphism is a root isomorphism.)

We give now an explicit description of the classical simple ind-groups $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$ whose Lie algebras are respectively $\mathfrak{a}(\infty)$, $\mathfrak{b}(\infty)$, $\mathfrak{c}(\infty)$, and $\mathfrak{d}(\infty)$.

$A(\infty)$. Here $G = A(\infty) = \varinjlim A(n-1)$ where $G_n = A(n-1) = SL(n)$ and the inclusion $A(n-1) \subset A(n)$ is given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Fixing the Cartan subalgebra $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g} = \mathfrak{a}(\infty)$, we have $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i \in \mathfrak{h}^*$ is given by $\varepsilon_i(\operatorname{diag}\{t_1, t_2, \dots\}) = t_i$. The Weyl group $\mathcal{W}$ consists of all permutations of $\{\varepsilon_i\}$ which leave all but finitely many ε_i fixed.

$B(\infty)$. Here $G = B(\infty) = \varinjlim B(n)$, where $B(n) = SO(2n+1)$ is the complex special orthogonal group corresponding to the nondegenerate symmetric bilinear form $(u, v) = u_1 v_1 + \sum_1^n (u_{2i} v_{2i+1} + u_{2i+1} v_{2i})$ on $\mathbb{C}^{2n+1}$, and where the inclusion $B(n) \subset B(n+1)$ is given by $g \mapsto \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Fixing the Cartan subalgebra $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g} = \mathfrak{b}(\infty)$, we have $\Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i \in \mathfrak{h}^*$ is given by $\varepsilon_i(\operatorname{diag}\{0, t_1, -t_1, t_2, -t_2, \dots\}) = t_i$. The Weyl group $\mathcal{W}$ consists of all signed permutations of $\{\varepsilon_i\}$ which leave all but finitely many ε_i fixed.

$C(\infty)$. Here $G = C(\infty) = \varinjlim C(n)$, where $C(n) = Sp(2n)$ is the complex symplectic group corresponding to the nondegenerate antisymmetric bilinear form $\langle u, v\rangle = \sum_1^n (u_{2i-1} v_{2i} - u_{2i} v_{2i-1})$ on $\mathbb{C}^{2n}$, and where the inclusion $C(n) \subset C(n+1)$ is given by $g \mapsto \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Fixing the Cartan subalgebra $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g} = \mathfrak{c}(\infty)$, we have $\Delta = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i \in \mathfrak{h}^*$ is given by $\varepsilon_i(\operatorname{diag}\{t_1, -t_1, t_2, -t_2, \dots\}) = t_i$. The Weyl group $\mathcal{W}$ consists of all signed permutations of $\{\varepsilon_i\}$ which leave all but finitely many ε_i fixed.

$D(\infty)$. Here $G = D(\infty) = \varinjlim D(n)$, where $D(n) = SO(2n)$ is the complex special orthogonal group corresponding to the nondegenerate symmetric bilinear form $(u, v) = \sum_1^n (u_{2i-1} v_{2i} + u_{2i} v_{2i-1})$ on $\mathbb{C}^{2n}$, and where the inclusion $D(n) \subset D(n+1)$ is given by $g \mapsto \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Fixing the Cartan subalgebra $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g} = \mathfrak{d}(\infty)$, we have $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$ where $\varepsilon_i \in \mathfrak{h}^*$ is given by $\varepsilon_i(\operatorname{diag}\{t_1, -t_1, t_2, -t_2, \dots\}) = t_i$. The Weyl group $\mathcal{W}$ consists of all signed permutations of $\{\varepsilon_i\}$ which leave all but finitely many ε_i fixed.

If $\mathfrak{g}$ is a root-reductive direct limit Lie algebra and $\mathfrak{k}$ is a root subalgebra, we set $\Delta_{\mathfrak{k}}^{\mathrm{ss}} := \Delta_{\mathfrak{k}} \cap (-\Delta_{\mathfrak{k}})$ (where $\Delta_{\mathfrak{k}}$ denotes the set of roots of $\mathfrak{k}$) and define $\mathfrak{k}^{\mathrm{ss}}$ to be the Lie subalgebra of $\mathfrak{k}$ generated by $\bigoplus_{\alpha \in \Delta_{\mathfrak{k}}^{\mathrm{ss}}} \mathfrak{k}^{\alpha}$.

The Lie algebra part of the following Proposition 4.1 reformulates [8, Theorem 1]. It describes the relationship between an arbitrary root-reductive direct limit Lie algebra and the classical simple direct limit Lie algebras. The group level statements follow from the algebra level statements.

PROPOSITION 4.1. *Let G be a root-reductive ind-group and $\mathfrak{g}$ be its Lie algebra.*

(i) $\mathfrak{g} = \mathfrak{g}^{\mathrm{ss}} \subsetplus \mathfrak{a}$ *for some abelian Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Furthermore, $\mathfrak{h} = \mathfrak{h}^{\mathrm{ss}} \oplus \mathfrak{a}$ where $\mathfrak{h}^{\mathrm{ss}} := \mathfrak{h} \cap \mathfrak{g}^{\mathrm{ss}}$.*

(ii) *G has a connected closed normal ind-subgroup G^{ss} with Lie algebra $\mathfrak{g}^{\mathrm{ss}}$ and a connected abelian ind-subgroup A with Lie algebra $\mathfrak{a}$, and $(g,a) \mapsto ga$ defines a homomorphism of the semidirect product $G^{\mathrm{ss}} \rtimes A$ onto G with discrete kernel Z. Furthermore $H \cong (H^{\mathrm{ss}} \times A)/Z$ where $H^{\mathrm{ss}} := H \cap G^{\mathrm{ss}}$ has Lie algebra $\mathfrak{h}^{\mathrm{ss}}$.*

(iii) *$\mathfrak{g}^{\mathrm{ss}} \cong \bigoplus_{t \in T} \mathfrak{g}^t$ where the $\mathfrak{g}^t$ are classical simple direct limit algebras or simple finite dimensional Lie algebras. $G^{\mathrm{ss}} \cong (\overset{\mathrm{f}}{\times}_{t \in T} G^t)/Z_1$ where G^t is the connected ind-group of G with Lie algebra $\mathfrak{g}^t$ and where Z_1 is a discrete central subgroup of G^{ss}.*

Example 4.2. Set $GL(\infty) := \varinjlim GL(n)$ where the inclusion $GL(n) \hookrightarrow GL(n{+}1)$ is given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Denote $A := \{\mathrm{diag}\,\{a, 1, 1, 1, \ldots\} \mid a \in \mathbb{C}^{\times}\} \cong \mathbb{C}^{\times}$. Then $GL(\infty) \cong SL(\infty) \rtimes A$ under $g \mapsto (g\alpha^{-1}, \alpha)$ where $\alpha := \mathrm{diag}\,\{\det(g), 1, 1, 1, \ldots\}$.

Throughout this paper, when considering a root-reductive Lie algebra $\mathfrak{g}$ we fix a Cartan subalgebra $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$ corresponding to a fixed system of root injections $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$. Without loss of generality we assume that any Borel or parabolic subalgebras of $\mathfrak{g}$ are chosen so that they contain $\mathfrak{h}$. Parabolic subgroups (including Borel subgroups) of a root-reductive ind-group G will thus contain the Cartan subgroup $H = \bigcup_n H_n$ with Lie algebra $\mathfrak{h}$. When we refer to $A(\infty)$, $B(\infty)$, $C(\infty)$, $D(\infty)$ or $GL(\infty)$ (or to their Lie algebras) we will furthermore assume that the G_n (or $\mathfrak{g}_n$), as well as the Cartan subgroup H (respectively, the Cartan subalgebra $\mathfrak{h}$) are chosen precisely as in our above explicit description.

We now define the parabolic subgroup ${}^wP$ needed in the statement of Theorem 11.1 below. If P is a proper parabolic subgroup (containing H) of a root-reductive ind-group G and $\mathfrak{p}$ is the Lie algebra of P, then $\mathfrak{h}+\mathfrak{p}^{\mathrm{ss}}$ is a natural choice for $\mathfrak{p}^{\mathrm{red}}$. Let $w \in \mathcal{W}$ be a Weyl group element. We define ${}^w\mathfrak{p}$ to be the parabolic subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the $\mathfrak{h}$-root spaces $\mathfrak{g}^{\alpha}$ for $\alpha \in \Delta_{\mathfrak{p}} \cup w(\Delta_{\mathfrak{p}^{\mathrm{red}}})$. Then ${}^wP$ is the parabolic subgroup of G with Lie algebra ${}^w\mathfrak{p}$. The subgroup ${}^wP$ is not necessarily proper, as shown by the following example.

Example 4.3. Let $G = GL(\infty)$, let $\mathfrak{p} \subset \mathfrak{gl}(\infty)$ be the (maximal) parabolic subalgebra with roots $\{\varepsilon_1 - \varepsilon_i \mid 2 \leq i\} \cup \{\varepsilon_i - \varepsilon_j \mid 2 \leq i \neq j\}$, and let $P \subset G$ be the corresponding parabolic subgroup. If $w = \sigma_\alpha$ for $\alpha = \varepsilon_1 - \varepsilon_2$, then ${}^wP = G$.

Next we describe the parabolic and Borel subalgebras of root-reductive direct limit Lie algebras $\mathfrak{g}$. In view of Proposition 4.1 it suffices to describe parabolic and Borel subalgebras for classical simple $\mathfrak{g}$. This gives also a description of the parabolic and Borel subgroups of root-reductive ind-groups G, for if P is a parabolic subgroup of G with Lie algebra $\mathfrak{p}$, then $P = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{p} = \mathfrak{p}\}$. The following statement, which of course is standard in the finite dimensional case, is a variation of Proposition 5 from [8].

PROPOSITION 4.4. *Suppose either that $\mathfrak{g}$ is a classical simple direct limit Lie algebra or that $\mathfrak{g} \cong \mathfrak{gl}(\infty)$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Then, for some index set $S_\mathfrak{p}$, there is an isomorphism of Lie algebras*

$$\mathfrak{p}^{\mathrm{red}} \cong \bigoplus_{s \in S_\mathfrak{p}} \mathfrak{l}^s, \tag{4.5}$$

where each $\mathfrak{l}^s$ is a subalgebra of $\mathfrak{p}^{\mathrm{red}}$ isomorphic to $\mathfrak{gl}(n)$, to $\mathfrak{gl}(\infty)$, to a finite dimensional simple Lie algebra, or to a classical simple direct limit Lie algebra. Furthermore, if $\mathfrak{g}$ is classical simple $S_\mathfrak{p}$ can be chosen so that $\mathfrak{l}^s$ is isomorphic to $\mathfrak{gl}(n)$ or $\mathfrak{gl}(\infty)$ for any $s \in S_\mathfrak{p}$ except for at most one index $s_0 \in S_\mathfrak{p}$ for which $\mathfrak{l}^{s_0}$ is isomorphic to:

$\mathfrak{a}(n)$ or $\mathfrak{a}(\infty)$ for $\mathfrak{g} = \mathfrak{a}(\infty)$;	$\mathfrak{a}(n), \mathfrak{a}(\infty), \mathfrak{b}(n)$ or $\mathfrak{b}(\infty)$ for $\mathfrak{g} = \mathfrak{b}(\infty)$;
$\mathfrak{a}(n), \mathfrak{a}(\infty), \mathfrak{c}(n)$ or $\mathfrak{c}(\infty)$ for $\mathfrak{g} = \mathfrak{c}(\infty)$;	$\mathfrak{a}(n), \mathfrak{a}(\infty), \mathfrak{d}(n)$ or $\mathfrak{d}(\infty)$ for $\mathfrak{g} = \mathfrak{d}(\infty)$.

For a classical simple $\mathfrak{g}$, there is a natural choice for the set $S_\mathfrak{p}$, and moreover $S_\mathfrak{p}$ is linearly ordered. Indeed, fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{p}$. For every n the parabolic subalgebra $\mathfrak{p}_n = \mathfrak{g}_n \cap \mathfrak{p}$ determines "marked" nodes in the Dynkin diagram of $\mathfrak{g}_n$ which correspond to simple roots α such that both α and $-\alpha$ are roots of $\mathfrak{p}_n$. Let $S_{\mathfrak{p}_n}$ be the set whose elements are all the unmarked nodes together with all connected components of marked nodes. We fix an order on the nodes of the Dynkin diagram of $\mathfrak{g}_n$ which is increasing from left to right. For $\mathfrak{g} \neq \mathfrak{d}(\infty)$ this order is unique. For $\mathfrak{g} = \mathfrak{d}(\infty)$, if the two rightmost nodes are both marked or both unmarked, we set the upper one to precede the lower one, otherwise, we set the marked one to precede the unmarked one. This order on the Dynkin diagram induces an order on $S_{\mathfrak{p}_n}$, and it is straightforward to check that the orders on $S_{\mathfrak{p}_n}$ are compatible for different n. Hence they determine a linear order on the union $\bigcup_n S_{\mathfrak{p}_n}$. In what follows, $S_\mathfrak{p}$ will be fixed as the union $\bigcup_n S_{\mathfrak{p}_n}$. Note that the order on $S_\mathfrak{p}$ depends only on $\mathfrak{p}$ and not on $\mathfrak{b}$. Furthermore, if $s_0 \in S_\mathfrak{p}$ and $\mathfrak{l}^{s_0}$ is not isomorphic to $\mathfrak{gl}(\infty)$, $\mathfrak{gl}(n)$, $\mathfrak{a}(\infty)$ or $\mathfrak{a}(n)$, then s_0 is necessarily the unique maximal element of $S_\mathfrak{p}$.

5. Diagonal ind-groups and beyond. The class of root-reductive ind-groups is part of the more general class of diagonal ind-groups. The corresponding class of Lie algebras has been studied quite extensively; see [2], [4], [1] and [20]. An essential difference between root-reductive direct limit Lie algebras and general diagonal direct limit Lie algebras is that the latter need not admit Cartan subalgebras that yield a root decomposition compatible with the direct limit. In this paper we do not develop a complete Bott-Borel-Weil theory for diagonal ind-groups.

Here is a diagonal ind-group that is not root-reductive. Consider the sequence of closed immersions

$$GL(2^n) \hookrightarrow GL(2^{n+1}), \qquad g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix},$$

and let $GL(2^\infty)$ denote the corresponding ind-group. Its Lie algebra $\mathfrak{gl}(2^\infty)$ is a diagonal direct limit Lie algebra [1]. If $H = \varinjlim H_n$ where H_n denotes the diagonal matrices in $GL(2^n)$, then H is a Cartan subgroup of $GL(2^\infty)$ and $\mathfrak{gl}(2^\infty)$ has no root decomposition with respect to $\mathfrak{h}$.

Let $\Delta_n = \{\varepsilon_{i,n} - \varepsilon_{j,n} \mid 1 \le i,j \le 2^n\}$ be the root system of $\mathfrak{gl}(2^n)$. The Borel subalgebras of $\mathfrak{gl}(2^\infty)$ containing $\mathfrak{h}$ are in bijective correspondence with the systems of triangular decompositions $\Delta_n = \Delta_n^+ \sqcup \Delta_n^-$ satisfying the following compatibility condition: if $\varepsilon_{i,n} - \varepsilon_{j,n} \in \Delta_n^+$, then $\varepsilon_{i,n+1} - \varepsilon_{j,n+1} \in \Delta_{n+1}^+$ and $\varepsilon_{2^n+i,n+1} - \varepsilon_{2^n+j,n+1} \in \Delta_{n+1}^+$. One can give a similar description of all parabolic subalgebras containing $\mathfrak{h}$ in terms of compatible systems of partitions $\Delta_n = \Delta_n^+ \sqcup \Delta_n^0 \sqcup \Delta_n^-$.

Here is another interesting ind-group. Its Lie algebra was introduced in [1]. Fix $k \in \mathbb{N}$, $k > 1$. Let the κ_n: $PGL(k^{2^n}) \hookrightarrow PGL(k^{2^{n+1}})$ be the unique closed immersions of algebraic groups for which the diagrams

$$\begin{array}{ccc} GL(k^{2^n}) & \xrightarrow{\mathrm{Ad}} & GL(k^{2^{n+1}}) \\ pr_n \downarrow & & \downarrow pr_{n+1} \\ PGL(k^{2^n}) & \overset{\kappa_n}{\hookrightarrow} & PGL(k^{2^{n+1}}) \end{array}$$

are commutative, where pr_n and pr_{n+1} are the natural projections. We define the reductive ind-group $PGL^{\mathrm{Ad}}(k^{2^\infty}) := \varinjlim PGL(k^{2^n})$. If $H_n \subset PGL(k^{2^n})$ denotes the subgroup of diagonal matrices, then $\kappa_n(H_n) \subset H_{n+1}$ and $H := \varinjlim H_n$ is a Cartan subgroup of $PGL^{\mathrm{Ad}}(k^{2^\infty})$.

A Borel subgroup $B \subset PGL^{\mathrm{Ad}}(k^{2^\infty})$ is determined by the $B_n := B \cap PGL(k^{2^n})$. Given B_n, we describe the Borel subgroups of $PGL(k^{2^{n+1}})$ that contain H_{n+1} and $\kappa_n(B_n)$, providing recursive descriptions of all Borel subgroups of $PGL^{\mathrm{Ad}}(k^{2^\infty})$ that contain H. Note first that specification of B_{n+1} is the same as specification of a $pr_{n+1}^{-1}(H_{n+1})$-invariant maximal flag in the natural representation space of $GL(k^{2^{n+1}})$, and that natural representation space is the adjoint representation space for $GL(k^{2^n})$. Hence, a $pr_{n+1}^{-1}(H_{n+1})$-invariant maximal flag in the natural representation of $GL(k^{2^{n+1}})$ is determined by a linear order on root basis of $\mathfrak{gl}(k^{2^n})$, i.e. a basis

$$\{x_\alpha\}_{\alpha\in\Delta_n} \cup \{h_i\}_{1\leq i\leq k^{2^n}} \tag{5.1}$$

consisting of root vectors x_α and of a basis $\{h_i\}_{1\leq i\leq k^{2^n}}$ of $\mathfrak{h}_n$, where Δ_n denotes the root system of $\mathfrak{gl}(k^{2^n})$. The Borel subgroup B_n determines a partition $\Delta_n = \Delta_n^+ \sqcup \Delta_n^-$ and thus also the following partial order $>$ on Δ_n: for $\alpha, \beta \in \Delta_n^+$, $\alpha > \beta$ whenever the $\mathfrak{b}_n$-height of α is greater then the $\mathfrak{b}_n$-height of β; $\alpha > \beta$ for any $\alpha \in \Delta_n^+$ and any $\beta \in \Delta_n^-$; and finally, for $\alpha, \beta \in \Delta_n^-$, $\alpha > \beta$ whenever $-\beta > -\alpha$. Now, since Δ_n is naturally identified with the set $\{x_\alpha\}_{\alpha\in\Delta_n}$, we can consider all extensions of this partial order on Δ_n to a linear order $>$ on the root basis (5.1) such that $x_\alpha > h_i > x_\beta$ for all i whenever $\alpha \in \Delta_n^+$ and $\beta \in \Delta_n^-$. Any such extension determines a unique $pr_{n+1}^{-1}(H_{n+1})$-invariant maximal flag in the natural representation and thus a unique Borel subgroup of $PGL(k^{2^{n+1}})$. One can check that the Borel subgroups obtained in this way are precisely the Borel subgroups of $PGL(k^{2^{n+1}})$ that contain H_{n+1} and $\kappa_n(B_n)$.

Part II. Representations.

6. Rational and pro-rational G-modules. Let $G = \varinjlim G_n$ be an ind-group. We define a *G-module* to be a vector space V endowed with G_n-module structures φ^n: $G_n \times V \to V$ ($\mathbb{C}$-linear in V) such that $\varphi^{n+1}|_{G_n\times V} = \varphi^n$ for all n. The maps φ^n define the *structure map* φ: $G \times V \to V$ of the G-module V. We say that V is a *rational G-module* if in addition the dimension of V is countable and φ is a morphism of ind-varieties, where V has the canonical ind-variety structure of Example 1.2. An equivalent definition of a rational G-module: V is isomorphic to the limit of a direct system of injections of rational finite dimensional G_n-modules φ_n: $G_n \times V_n \to V_n$. Every rational G-module is *locally finite*, in other words, $\varphi(G_n \times \mathbb{C}v)$ generates a finite dimensional submodule for every $v \in V$ and every n.

Any rational module over a reductive algebraic group is completely reducible. Thus, if G is locally reductive and $V = \varinjlim V_n$ is a rational G-module, then V is a completely reducible G_n-module for every n.

The category of rational G-modules is too restrictive for our purposes. For example the dual of a rational G-module is no longer rational (as $\dim V^*$ is in general uncountable). We define a G-module U to be *pro-rational* if it is the dual of a rational G-module. This is equivalent to saying that U is isomorphic to the inverse (or projective) limit $\varprojlim U_n$ of a system

$$\cdots \xrightarrow{\psi_{n+1}} U_n \xrightarrow{\psi_n} U_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_1} U_0 \longrightarrow 0$$

of finite dimensional rational G_n-modules U_n, where ψ_n is a rational G_{n-1}-module surjection for each n. The G-module structure on $\varprojlim U_n$: if $u = (\dots, u_k, u_{k-1}, \dots, u_0) \in \varprojlim U_n$, so each $u_n \in U_n$ and $\psi_n(u_n) = u_{n-1}$ for $n \geq 1$, and if $g_m \in G_m$, then

$$g_m \cdot u := (\dots, g_m \cdot u_k, g_m \cdot u_{k-1}, \dots, g_m \cdot u_m, \psi_m(g_m \cdot u_m), \dots, \psi_1 \circ \cdots \circ \psi_m(g_m \cdot u_m))$$

where $k > m$.

7. $\mathfrak{g}$-modules. If G is an ind-group and V is a rational G-module with structure map φ: $G \times V \to V$, then V is a module for the Lie algebra $\mathfrak{g}$ of G with structure map $d\varphi$: $\mathfrak{g} \times V \to V$. We say that a $\mathfrak{g}$-module V is *rationally G-integrable* (or, briefly, *G-integrable*) if it is obtained by this construction from a rational G-module structure on V. As G is assumed to be connected, that G-module structure is unique. Furthermore, a G-integrable $\mathfrak{g}$-module V necessarily is *locally finite*. By definition this means that V is locally finite as $\mathfrak{g}_n$-module for every n, in other words, $\dim \mathcal{U}(\mathfrak{g}_n) \cdot v < \infty$ for every $v \in V$ and every n. (In some works on Lie algebra representations, in particular in [15] and [8], the term “integrable” is a synonym for various versions of local finiteness. This is not acceptable in the present paper because such representations need not integrate from the Lie algebra to the ind-group.) It is straightforward to verify that a countable-dimensional $\mathfrak{g}$-module V is locally finite if and only if V is isomorphic to a direct limit $\varinjlim V_n$ of finite dimensional $\mathfrak{g}_n$-modules V_n. Unless the contrary is stated explicitly, in what follows we will assume automatically that an expression of a locally finite $\mathfrak{g}$-module as $\varinjlim V_n$ corresponds to a direct system of injections $V_n \hookrightarrow V_{n+1}$.

In the rest of this section, G is a locally reductive ind-group, $\mathfrak{g}$ is its Lie algebra, and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. We study irreducible locally finite $\mathfrak{p}$-modules and parabolically generated irreducible $\mathfrak{g}$-modules.

PROPOSITION 7.1. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let E be an irreducible locally finite $\mathfrak{p}$-module and let $\mathfrak{u}_\mathfrak{p}$ denote the nilpotent radical of $\mathfrak{p}$. Then $\mathfrak{u}_\mathfrak{p} \cdot E = 0$.*

Proof. Suppose that $\xi \cdot e \neq 0$ for some $\xi \in \mathfrak{u}_\mathfrak{p}$ and $e \in E$. As E is irreducible there exists $u \in \mathcal{U}(\mathfrak{p})$ such that $u \cdot \xi \cdot e = e$. Let n be sufficiently large so that $\xi, u \in \mathcal{U}(\mathfrak{p}_n)$. Then $E'_n := \mathfrak{u}_{\mathfrak{p}_n} \cdot E_n$ is a proper $\mathfrak{p}_n$-submodule of $E_n := \mathcal{U}(\mathfrak{p}_n) \cdot e$ such that $e \notin E'_n$. But as $\xi \cdot e \in E'_n$, the equality $u \cdot \xi \cdot e = e$ is contradictory. Therefore $\xi \cdot e$ cannot be nonzero. □

COROLLARY 7.2. *If P is a parabolic subgroup of G and E is an irreducible rational P-module, then the unipotent radical U_P of P acts trivially on E.*

Let E be an irreducible $\mathfrak{p}$-module as above. We introduce the induced $\mathfrak{g}$-module

$$\tilde{V}(E) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E.$$

If $\mathfrak{p}$ is a Borel subalgebra then $\dim E = 1$, i.e., $E = \mathbb{C}_\mu$ for some $\mu \in \mathfrak{h}^*$, and $\tilde{V}(\mathbb{C}_\mu)$ is called a Verma module. In general, $\dim E = \infty$, but nevertheless the following proposition holds. It is very similar to Theorem 2.1 in [1] and therefore we omit the proof.

PROPOSITION 7.3. *Let E be an irreducible locally finite $\mathfrak{p}$-module. Then $\tilde{V}(E)$ has a unique maximal proper $\mathfrak{g}$-submodule I_E.*

An important corollary of Proposition 7.3 is that $\tilde{V}(E)$ has a unique irreducible quotient module $V(E) := \tilde{V}(E)/I_E$. We call $V(E)$ the *irreducible $\mathfrak{g}$-module parabolically generated by E*. Any $\mathfrak{g}$-module V which is generated by a $\mathfrak{p}$-submodule isomorphic to E has the $\mathfrak{g}$-module $V(E)$ as a quotient. Indeed, V admits an obvious surjection $\tilde{\sigma}$: $\tilde{V}(E) \to V$ whose kernel is necessarily a $\mathfrak{g}$-submodule of I_E. Thus $\tilde{\sigma}$ induces a $\mathfrak{g}$-surjection s: $V = (\tilde{V}(E)/\ker\tilde{\sigma}) \to V(E) = \tilde{V}(E)/I_E$.

In general, $V(E)$ is not locally finite, and therefore is not rationally G-integrable. The problem of characterizing, for a fixed $\mathfrak{p}$, all irreducible locally finite $\mathfrak{p}$-modules E for which $V(E)$ is G-integrable, is a generalization of the problem of computing all dominant integral weights for a finite dimensional reductive group. The following Proposition reduces this problem in an explicit way to the structure of E. The case when $\mathfrak{p}$ is a Borel subalgebra of a classical simple linear Lie algebra was studied in [1] and [21].

PROPOSITION 7.4. *Let E be an irreducible locally finite $\mathfrak{p}$-module. Then $V(E)$ is G-integrable if and only if E is P-integrable and, for any expression $E = \varinjlim E_n$, $V(E_n)$ is a finite dimensional $\mathfrak{g}_n$-module for each n. When these conditions hold, each $V(E_n)$ is a well-defined G_n-module and we have a canonical G-module isomorphism $V(E) \cong \varinjlim V(E_n)$.*

Proof. First, if E is P-integrable and $V(E_n)$ is finite dimensional for each n, the standard theory of connected algebraic groups applied to G_n implies that $V(E_n)$ is G_n-integrable for any n. Furthermore, $\varinjlim V(E_n)$ is a rational G-module which admits a P-module injection i: $E \hookrightarrow \varinjlim V(E_n)$ such that $i(E)$ generates $\varinjlim V(E_n)$ as a G-module. This is sufficient to conclude that $\varinjlim V(E_n) = V(E)$. Indeed, one need only check that the $\mathfrak{g}$-surjection s: $\varinjlim V(E_n) \to V(E)$ induced by i is an isomorphism. Assuming that $\ker s \neq 0$ we find an n such that $\ker s \cap V(E_n) \neq 0$. Then, as $V(E_n)$ is an irreducible $\mathfrak{g}_n$-module, $V(E_n) \subset \ker s$. Therefore $\ker s \cap i(E_n) \neq 0$, which contradicts the injectivity of i.

Conversely, if $V(E)$ is G-integrable, E must be P-integrable. Indeed, if $E = \varinjlim E_n$, then for each n E_n is a $\mathfrak{p}_n$-submodule of the finite dimensional G_n-submodule of $V(E)$ generated by E_n. Therefore, again the theory of connected algebraic groups implies that E_n is necessarily P_n-integrable. Thus $\varinjlim E_n$ is a P-integrable $\mathfrak{p}$-module. To complete the proof we need to show also that $V(E_n)$ is finite dimensional for each n (and is thus a G_n-module) whenever $V(E)$ is G-integrable. This follows from a standard geometric version of Frobenius

Reciprocity and we present this argument in Section 12 in the proof of Theorem 11.1(i). □

In the following, we say that an irreducible rational P-module E is *dominant* if $V(E)$ is a locally finite $\mathfrak{g}$-module, and thus is a well-defined irreducible rational G-module. An integral weight $\lambda \in \mathfrak{h}^*$ is *B-dominant* for a Borel subgroup $B \subset G$ if the one dimensional B-module $\mathbb{C}_\lambda$ of weight λ is dominant.

We conclude this Section by recalling some basic definitions for weight modules. Let $\mathfrak{g}$ be root-reductive. A $\mathfrak{g}$-module V is a *weight module* if it has an $\mathfrak{h}$-module decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V^\mu, \tag{7.5}$$

where $\mathfrak{h}$ is the fixed Cartan subalgebra of $\mathfrak{g}$ and $V^\mu := \{v \in V \mid h \cdot v = \mu(h)v \text{ for any } \mu \in \mathfrak{h}^*\}$. The *support* $\operatorname{supp} V$ of a weight module V is $\{\mu \in \mathfrak{h}^* \mid V^\mu \neq 0\}$. A weight $\mathfrak{g}$-module V is *finite* if the support of V is finite in the direction of every root of $\mathfrak{g}$, i.e., if for every $\mu \in \operatorname{supp} V$ and every $\alpha \in \Delta$, the intersection $\{\mu + k\alpha \mid k \in \mathbb{Z}_+\} \cap \operatorname{supp} V$ is finite. See [8]. A finite $\mathfrak{g}$-module is locally finite. In the following we will consider finite $\mathfrak{g}$- and $\mathfrak{p}$-modules, the latter being defined as $\mathfrak{p}$-modules which are weight modules (i.e., which satisfy (7.5)) and which are finite as $\mathfrak{p}^{\mathrm{red}}$-modules. Also, we define a rational G- or P-module to be *finite* if it is finite respectively as a weight $\mathfrak{g}$- or $\mathfrak{p}$-module.

8. Irreducible rational G-modules. Let G be a reductive ind-group. An irreducible rational G- (or P-) module V is called *locally irreducible* if $V = \varinjlim V_n$ for some direct system of irreducible rational (finite dimensional) G_n- (or P_n-) modules V_n. Our starting point in this section is that an irreducible rational G-module V is not necessarily locally irreducible. Here are some examples.

Example 8.1. For every n fix a pair of nonisomorphic irreducible finite dimensional G_n-modules U_n, W_n such that $U_n, W_n \prec U_{n+1}$, and $U_n, W_n \prec W_{n+1}$, where here the sign $\prec$ indicates the existence of a G_n-module injection. Extend the diagonal injection $U_n \hookrightarrow U_n \oplus U_n$ to a G_n-module injection $\eta_n^U\colon U_n \hookrightarrow U_{n+1} \oplus W_{n+1}$. Similarly, fix a G_n-module injection $\eta_n^W\colon W_n \hookrightarrow U_{n+1} \oplus W_{n+1}$. Define $\eta_n\colon U_n \oplus W_n \hookrightarrow U_{n+1} \oplus W_{n+1}$ by $\eta_n := \eta_n^U \oplus \eta_n^W$. Let $V_n := U_n \oplus W_n$. Then the G-module $V = \varinjlim V_n$ is an irreducible rational G-module that is not locally irreducible. It is irreducible as a consequence of Proposition 8.3 below; see Example 8.4(1). It is not locally irreducible because, for every nonzero $v \in V$ and for sufficiently large n, the G_n-submodule of V generated by v is isomorphic to $U_n \oplus W_n$, and thus is reducible.

Here is an explicit choice of the G_n-modules U_n and W_n. Let $G = GL(\infty)$. Let U_n and W_n be the respective irreducible $GL(n)$-modules with highest weights $\lambda_n := \varepsilon_1 - (n-1)\varepsilon_n$ and $\mu_n := \varepsilon_1 - n\varepsilon_n$. The standard branching rule [9] ensures that $U_n, W_n \prec U_{n+1}$, and $U_n, W_n \prec W_{n+1}$. For this particular choice of U_n and W_n the resulting module V is not a weight module. Other choices of U_n and W_n yield irreducible weight $GL(\infty)$-modules that are irreducible but not locally irreducible.

Now we introduce an essential invariant of any locally finite $\mathfrak{g}$-module V. Represent V as the limit of a direct system of finite dimensional semisimple $\mathfrak{g}_n$-modules,

$$0 \longrightarrow V_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} V_n \xrightarrow{\varphi_n} V_{n+1} \xrightarrow{\varphi_{n+1}} \cdots, \tag{8.2}$$

where the φ_i need not be injective. Each V_n decomposes canonically as a direct sum of its isotypic components, $V_n = \bigoplus_i V_n^i$, and we fix decompositions $V_n^i = \bigoplus_k V_n^{i,k}$ into simple $\mathfrak{g}_n$-modules. To each $V_n^{i,k}$ we assign an abstract vector $v_n^{i,k}$ and define a vector space $\mathcal{V}_n^i := \bigoplus_k \mathbb{C} v_n^{i,k}$. When the composition of $\varphi_n|_{V_n^i}$ with the projection $V_{n+1} \to V_{n+1}^j$ is nonzero, we define a linear map

$$\alpha_{i,j}\colon\ V_n^i \to V_{n+1}^j \text{ by } \alpha_{i,j}(v_n^{i,k}) := \sum\nolimits_{k'} v_{n+1}^{j,k'},$$

where k' runs over the simple components of V_{n+1}^j onto which $\varphi(V_n^{i,k})$ projects nontrivially. The collection $\{\mathcal{V}_n^i, \alpha_{i,j}\}$ is, by definition, a *multiplicity diagram* D_V of V. D_V is a commutative diagram of finite dimensional vector spaces.

A *subdiagram* of a multiplicity diagram $D_V = \{\mathcal{V}_n^i, \alpha_{i,j}\}$ is a collection $D' = \{(\mathcal{V}_n^i)', \alpha'_{i,j}\}$ of subspaces $(\mathcal{V}_n^i)' \subset \mathcal{V}_n^i$ and linear maps $\alpha'_{i,j}\colon\ (\mathcal{V}_n^i)' \to (\mathcal{V}_{n+1}^j)'$, where $\alpha'_{i,j}$ is simply the restriction of $\alpha_{i,j}$ to $(\mathcal{V}_n^i)'$. A subdiagram D' of D_V is *stably proper* if there is no index n_0 such that $(V_n^i)' = V_n^i$ for all $n \geq n_0$ and all i. Finally, we call a multiplicity diagram D_V *minimal* (and call the direct system (8.2) *minimal*) if no stably proper subdiagram D' of D_V is a multiplicity diagram of V.

The following is a straightforward but important proposition.

PROPOSITION 8.3. *Let V be a locally finite $\mathfrak{g}$-module.*

(i) *If V is finitely generated (in particular, if V is irreducible), V admits a minimal multiplicity diagram D_V.*

(ii) *If V admits a multiplicity diagram which has no nonzero stably proper subdiagrams, then V is irreducible.*

(iii) *If V is irreducible and D_V is a minimal multiplicity diagram of V, then D_V has no nonzero stably proper subdiagrams.*

Proof. (i) Fix a finite dimensional subspace $\bar{V} \subset V$ which generates V as a $\mathfrak{g}$-module and set V_n to be the $\mathfrak{g}_n$-module generated by $\bar{V}$. Then $V = \varinjlim V_n$ is a minimal direct system of injections, and D_V is a minimal multiplicity diagram of V.

(ii) If D_V admits no nonzero stably proper subdiagram, V is necessarily generated by each "isotypic vector", i.e., by each $v \in V$ such that v is in the image of some $v' \in V_n^i$. This is sufficient to conclude that V is irreducible as any nonzero $\mathfrak{g}$-submodule of V intersects nontrivially with some nonzero $\mathfrak{g}_n$-isotypic component of V.

(iii) A nonzero stably proper subdiagram D' of D_V is immediately checked to give rise to a nonzero $\mathfrak{g}$-submodule $V' \subset V$ such that $D' = D_{V'}$. As D_V is minimal, V' is necessarily a proper submodule. Contradiction. $\square$

Example 8.4.

(1) If $V = \varinjlim V_n$, $V_n = U_n \oplus W_n$ as in the beginning of this section (in particular as in Example 8.1), then D_V has the form

$$\begin{tikzcd} \mathcal{V}_1^1 \arrow[r] \arrow[dr] & \mathcal{V}_2^1 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & \mathcal{V}_n^1 \arrow[r] \arrow[dr] & \cdots \\ \mathcal{V}_1^2 \arrow[r] \arrow[ur] & \mathcal{V}_2^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \mathcal{V}_n^2 \arrow[r] \arrow[ur] & \cdots, \end{tikzcd}$$

all spaces $\mathcal{V}_n^i$ being one dimensional and all maps being isomorphisms. Clearly, D_V is a minimal multiplicity diagram which has no nonzero stably proper subdiagrams. Therefore V is irreducible.

(2) Let $V_n := \mathfrak{gl}(n)$, $V_n^1 \cong \mathfrak{a}(n-1)$, $V_n^2 \cong \mathbb{C}$, and let $V_n \hookrightarrow V_{n+1}$ be the Lie algebra injection induced by the inclusion $GL(n) \hookrightarrow GL(n+1)$ of Example 4.2. Then $V \cong \mathfrak{gl}(\infty)$ and D_V has the form

$$\begin{tikzcd} \mathcal{V}_1^1 \arrow[r] & \mathcal{V}_2^1 \arrow[r] & \cdots \arrow[r] & \mathcal{V}_n^1 \arrow[r] & \cdots \\ \mathcal{V}_1^2 \arrow[r] \arrow[ur] & \mathcal{V}_2^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \mathcal{V}_n^2 \arrow[r] \arrow[ur] & \cdots, \end{tikzcd}$$

where all spaces $\mathcal{V}_n^i$ are one dimensional and all maps are isomorphisms. D_V is a minimal multiplicity diagram of V. Its upper row is a stably proper subdiagram which gives rise to the $\mathfrak{gl}(\infty)$-submodule $\mathfrak{a}(\infty) \subset \mathfrak{gl}(\infty)$.

Next we establish an important fact about the structure of parabolically generated irreducible locally finite $\mathfrak{g}$-modules which are weight modules.

PROPOSITION 8.5. *Let E be an irreducible dominant $\mathfrak{p}$-module which is a weight module. Assume furthermore that $\mathfrak{p}$ contains no entire simple component $\mathfrak{g}^t$ of $\mathfrak{g}^{\mathrm{ss}}$ (see Proposition 4.1). Then:*

(i) *E is a finite $\mathfrak{p}$-module and $V(E)$ is a finite $\mathfrak{g}$-module;*

(ii) *both E and $V(E)$ are locally irreducible.*

Proof. Proposition 6 in [8] implies that, for any $t' \in T$, $V(E)$ is a finite $\mathfrak{h} + \mathfrak{g}^{t'}$-module if and only if $\operatorname{supp} V(E)$ is finite in the direction of some root $\alpha \in \Delta^{t'}$. As $V(E)$ is immediately seen to be finite in the direction of any root $\alpha \in \Delta$ with $\mathfrak{g}^\alpha \subset \mathfrak{u}_\mathfrak{p}$, and as $\mathfrak{p}$ does not contain an entire simple component $\mathfrak{g}^t$, $V(E)$ is necessarily a finite $\mathfrak{g}$-module. In particular, $\operatorname{supp} E$ is finite in the direction of every root of $\mathfrak{p}^{\mathrm{red}}$, i.e., E is a finite $\mathfrak{p}$-module. Statement (i) is proved. Statement (ii) is a direct corollary of (i) and of Theorem 6(i) in [8]; the latter says that a finite module is locally irreducible. $\square$

The following example shows that local irreducibility in Proposition 8.5 really requires that E be a weight module.

Example 8.6. Let $G = GL(\infty)$, let P be as in Example 4.3, and let V be the G-module from Example 8.1. Set $E := \{v \in V \mid g \cdot v = v \text{ for all } g \in U_P\}$. Then a direct verification shows that E is an irreducible, but not locally irreducible, P-module and $V = V(E)$.

Concluding this section, we note that a locally irreducible rational G-module does not have to be a weight G-module. Here are some examples.

Example 8.7. Assume that G is root-reductive and fix an irreducible finite dimensional G_n-module V_n for every n, such that V_{n+1}, considered as a G_n-module, contains at least two irreducible components V_n' and V_n'' isomorphic to V_n whose supports are disjoint as subsets of $\operatorname{supp} V_{n+1}$. Define the G_n-injection $V_n \hookrightarrow V_{n+1}$ to be the composition of the diagonal injection of V_n into $V_n' \oplus V_n''$ and the injection of $V_n' \oplus V_n''$ into V_{n+1}. Then $V := \varinjlim V_n$ is a locally irreducible rational G-module which is not a weight module.

For an explicit example, let $G = A(\infty)$ and let V_n be the irreducible G_n-module with highest weight $\lambda_n := -\varepsilon_1 - 2\varepsilon_2 - \cdots - n\varepsilon_n$. Set $\lambda_{n+1}' := \lambda_{n+1}$ and $\lambda_{n+1}'' := -2\varepsilon_1 - 3\varepsilon_2 - \cdots - (n+1)\varepsilon_n - \varepsilon_{n+1}$. Both λ_{n+1}' and λ_{n+1}'' are extremal weights of V_{n+1}. The G_n-submodules of V_{n+1} generated by $V_{n+1}^{\lambda_{n+1}'}$ and $V_{n+1}^{\lambda_{n+1}''}$ are isomorphic to V_n and can be chosen as V_n' and V_n''.

9. Combinatorics of locally irreducible locally finite $\mathfrak{p}$-modules. Here we introduce and study certain combinatorial invariants of locally irreducible locally finite $\mathfrak{p}$-modules. Those invariants appear naturally in the geometric study of these modules. Fix a finite dimensional reductive Lie algebra $\mathfrak{k}$, a Cartan subalgebra $\mathfrak{h}_{\mathfrak{k}}$ and a parabolic subalgebra $\mathfrak{p}_{\mathfrak{k}}$ with $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{p}_{\mathfrak{k}} \subset \mathfrak{k}$. Denote the roots of $\mathfrak{k}$ by $\Delta_{\mathfrak{k}}$. Given a Borel subalgebra $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{p}_{\mathfrak{k}}$, we have a partition $\Delta_{\mathfrak{k}} = \Delta_{\mathfrak{k}}^+ \sqcup \Delta_{\mathfrak{k}}^-$. As usual, $\rho_{\mathfrak{b}_{\mathfrak{k}}}$ denotes the half-sum of the elements of $\Delta_{\mathfrak{k}}^+$, and $\operatorname{rk}\mathfrak{k}$ stands for the semisimple rank of $\mathfrak{k}$. If $E_{\mathfrak{k}}$ is an irreducible finite dimensional $\mathfrak{p}_{\mathfrak{k}}$-module, we call $E_{\mathfrak{k}}$ *regular* if $\lambda + \rho_{\mathfrak{b}_{\mathfrak{k}}}$ is a regular weight of $\mathfrak{k}$, where λ is the $\mathfrak{b}_{\mathfrak{k}}$-highest weight of $E_{\mathfrak{k}}$. To any $E_{\mathfrak{k}}$ we assign an integer $\ell(E_{\mathfrak{k}})$, its *length*, as follows. If $E_{\mathfrak{k}}$ is regular then $\ell(E_{\mathfrak{k}})$ is the length of the unique Weyl group element w for which $w(\lambda + \rho_{\mathfrak{b}_{\mathfrak{k}}})$ is $\mathfrak{b}_{\mathfrak{k}}$-dominant. If $E_{\mathfrak{k}}$ is not regular, we define $\ell(E_{\mathfrak{k}})$ inductively. If $\operatorname{rk}\mathfrak{k} = 0$, $\ell(E_{\mathfrak{k}}) := 0$. For $\operatorname{rk}\mathfrak{k} > 0$, we set $\ell(E_{\mathfrak{k}}) := \max\{\ell(E_{\mathfrak{k}'}) \mid E_{\mathfrak{k}'} \prec E_{\mathfrak{k}}\}$, where the maximum is taken over all modules $E_{\mathfrak{k}'} \prec E_{\mathfrak{k}}$ of all proper root subalgebras $\mathfrak{k}'$ of $\mathfrak{k}$.

LEMMA 9.1. *Let* $\mathfrak{k} \cong \mathfrak{a}(k-1)$ *and suppose that* $\mathfrak{s} \cong \mathfrak{a}(s-1)$ *is a root subalgebra of* $\mathfrak{k}$*, or let* $\mathfrak{k} \cong \mathfrak{gl}(k)$ *and suppose that* $\mathfrak{s} \cong \mathfrak{gl}(s)$ *is a root subalgebra of* $\mathfrak{k}$*. Set* $\mathfrak{p}_{\mathfrak{s}} := \mathfrak{p}_{\mathfrak{k}} \cap \mathfrak{s}$*, and fix an irreducible finite dimensional* $\mathfrak{p}_{\mathfrak{k}}$*-module* E_k *and an irreducible* $\mathfrak{p}_{\mathfrak{s}}$*-module* $E_s \prec E_k$*. Then:*

(i) $\ell(E_s) \leq \ell(E_k)$*;*

(ii) *if* $\ell(E_s) = \ell(E_k)$ *and both* E_s *and* E_k *are regular, there exists a chain of proper root subalgebras* $\mathfrak{s} \subset \mathfrak{k}_s \subset \cdots \subset \mathfrak{k}_m \subset \cdots \subset \mathfrak{k}_k = \mathfrak{k}$ *together with a chain of regular irreducible* $\mathfrak{p}_{\mathfrak{k}} \cap \mathfrak{k}_m$*-modules* E_m*,*

$$E_s \prec \cdots \prec E_m \prec \cdots \prec E_k.$$

Proof. (i) It suffices to prove that $\ell(E_s) \leq \ell(E_k)$ when both E_k and E_s are regular. If E_k is not regular, the inequality follows from the definition of $\ell(E_k)$. If E_k is regular but E_s is not regular, then $\ell(E_s) = \ell(E_{s'})$ for some regular $E_{s'} \prec E_s$, and the inequality $\ell(E_{s'}) \leq \ell(E_k)$ gives $\ell(E_s) \leq \ell(E_k)$.

Assume $\Delta_{\mathfrak{k}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq k\}$ and $\Delta_{\mathfrak{p}_{\mathfrak{k}}} = (\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq k\}) \cup (\bigcup_{1 \leq q \leq a}\{-(\varepsilon_i - \varepsilon_j) \mid p_{q-1}+1 \leq i < j \leq p_q\})$ for $1 \leq p_1 < p_2 < \cdots < p_a = k$ with $p_0 = 0$. Then $\Delta_{\mathfrak{s}} = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I_s\}$ for some subset $I_s = \{r_1, \ldots, r_s\}$ of $I_k := \{1, \ldots, k\}$ where $r_1 < \cdots < r_s$. Fix a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{p}_{\mathfrak{k}}$ such that $\Delta^+_{\mathfrak{k}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq k\}$ and $[p_{q-1}+1, p_q] \cap \{r_1, \ldots, r_s\} = \{p_{q-1}+1, \ldots, p_{q-1}+b_q\}$ where b_q is the cardinality of $[p_{q-1}+1, p_q] \cap \{r_1, \ldots, r_s\}$. Define the chain of sets $I_s \subset \cdots \subset I_m \subset \cdots \subset I_k$ by $I_{m+1} := I_m \cup \{i_m\}$ where i_m is the smallest element of $I_k \backslash I_m$ for $s \leq m < k$. This determines a chain of root subalgebras $\mathfrak{s} = \mathfrak{k}_s \subset \cdots \subset \mathfrak{k}_m \subset \cdots \subset \mathfrak{k}_k = \mathfrak{k}$, $\mathfrak{k}_m \cong \mathfrak{a}(m-1)$ (resp. $\mathfrak{k}_m \cong \mathfrak{gl}(m)$) with $\Delta_{\mathfrak{k}_m} = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I_m\}$. Set $\Delta_m := \Delta_{\mathfrak{k}_m}$ and $\rho_m := \frac{1}{2}\sum_{\gamma \in \Delta^+_{\mathfrak{k}} \cap \Delta_m} \gamma$.

We define now a chain of irreducible $\mathfrak{p} \cap \mathfrak{k}_m$-modules E_m,

$$E_s \prec \cdots \prec E_m \prec \cdots \prec E_k,$$

by means of their $\mathfrak{b}_{\mathfrak{k}} \cap \mathfrak{k}_m$-highest weights λ_m. E_s and E_k are given. Expand λ_k and λ_s as $\lambda_k = \lambda_k^1\varepsilon_1 + \cdots + \lambda_k^k\varepsilon_k$ and $\lambda_s = \lambda_s^{r_1}\varepsilon_{r_1} + \cdots + \lambda_s^{r_s}\varepsilon_{r_s}$. Determine the integers $s = i_0 < \cdots < i_q < \cdots < i_a = k$ by the property that $\{1, 2, \ldots, p_q\} \subset I_{i_q}$ but $\{1, 2, \ldots, p_q\} \not\subset I_{i_q - 1}$, and set

$$\lambda_{i_q} := \lambda_k^1\varepsilon_1 + \cdots + \lambda_k^{p_q}\varepsilon_{p_q} + \lambda_s^{r_q}\varepsilon_{r_q} + \cdots + \lambda_s^{r_s}\varepsilon_{r_s}, \tag{9.2}$$

where r_q is the smallest integer in I_s such that $r_q > p_q$. Let $i_{q-1} < m < i_q$. Then $\lambda_{i_{q-1}} = \lambda_k^1\varepsilon_1 + \cdots + \lambda_k^{p_{q-1}}\varepsilon_{p_{q-1}} + \lambda_s^{p_{q-1}+1}\varepsilon_{p_{q-1}+1} + \cdots + \lambda_s^{p_{q-1}+b_q}\varepsilon_{p_{q-1}+b_q} + \lambda_s^{r_q}\varepsilon_{r_q} + \cdots + \lambda_s^{r_s}\varepsilon_{r_s}$ and $\lambda_{i_q} = \lambda_k^1\varepsilon_1 + \cdots + \lambda_k^{p_{q-1}}\varepsilon_{p_{q-1}} + \lambda_k^{p_{q-1}+1}\varepsilon_{p_{q-1}+1} + \cdots + \lambda_k^{p_{q-1}+(p_q-p_{q-1})}\varepsilon_{p_{q-1}+(p_q-p_{q-1})} + \lambda_s^{r_q}\varepsilon_{r_q} + \cdots + \lambda_s^{r_s}\varepsilon_{r_s}$. The standard branching rule gives

$$\lambda_k^{p_{q-1}+j} \geq \lambda_s^{p_{q-1}+j} \geq \lambda_k^{p_{q-1}+j+((p_q-p_{q-1})-b_q)} \tag{9.3}$$

for $1 \le j \le b_q$. For $1 \le j \le b_q + (m - i_{q-1})$, set

$$\lambda_m^{p_{q-1}+j} := \begin{cases} \lambda_k^{p_{q-1}+j} & \text{if} \quad j \le m - i_{q-1}, \\ \min\{\lambda_k^{p_{q-1}+j}, \lambda_s^{p_{q-1}+j-(m-i_{q-1})}\} & \text{if} \quad j > m - i_{q-1}. \end{cases} \tag{9.4}$$

Put $\lambda_m := \lambda_k^1\varepsilon_1 + \cdots + \lambda_k^{p_{q-1}}\varepsilon_{p_{q-1}} + \lambda_m^{p_{q-1}+1}\varepsilon_{p_{q-1}+1} + \cdots + \lambda_m^{p_{q-1}+b_q+(m-i_{q-1})}$ $\varepsilon_{p_{q-1}+b_q+(m-i_{q-1})} + \lambda_s^{r_q}\varepsilon_{r_q} + \cdots + \lambda_s^{r_s}\varepsilon_{r_s}$, and define E_m as the finite dimensional irreducible $\mathfrak{p}_\mathfrak{k} \cap \mathfrak{k}_m$-module with $\mathfrak{b}_\mathfrak{k} \cap \mathfrak{k}_m$-highest weight λ_m. Using (9.3) and (9.4), one verifies that $E_m \prec E_{m+1}$ for $i_{q-1} \le m < i_q$.

Note that the weights λ_m satisfy

$$\text{if } \alpha, \beta \in \Delta_\mathfrak{k}^+ \cap \Delta_m \text{ and } (\lambda_m + \rho_m, \alpha) = (\lambda_m + \rho_m, \beta) = 0, \text{ then } \alpha + \beta \notin \Delta_m. \tag{9.5}$$

Indeed, if $m = i_q$ for some $1 \le q \le a$, then (9.5) is immediate because $(\lambda_m + \rho_m, \alpha) = 0$ with $\alpha \in \Delta_\mathfrak{k}^+ \cap \Delta_m$ implies $\alpha = \varepsilon_x - \varepsilon_y$ for some $1 \le x \le p_q$ and $p_q < y \le k$. Assume now $i_{q-1} < m < i_q$. If $\alpha, \beta \in \Delta_\mathfrak{k}^+ \cap \Delta_m$ and $(\lambda_m + \rho_m, \alpha) = (\lambda_m + \rho_m, \beta) = 0$, one verifies (possibly after interchanging α and β) that $\alpha = \varepsilon_x - \varepsilon_y$ and $\beta = \varepsilon_y - \varepsilon_z$, where $1 \le x \le p_{q-1}$, $p_{q-1} < y \le p_q$ and $p_q < z \le k$. Then, using the definition of λ_m (i.e., formula (9.4)) one checks that, $\lambda_m^y = \lambda_k^y$ gives $(\lambda_m + \rho_m, \alpha) = (\lambda_k + \rho_\mathfrak{k}, \alpha) = 0$, i.e., a contradiction with the regularity of E_k. Similarly, $\lambda_m^y = \lambda_k^{y-(m-i_{q-1})}$ gives $(\lambda_m + \rho_m, \beta) = (\lambda_s + \rho_\mathfrak{s}, \beta) = 0$, which contradicts regularity of E_s. This establishes (9.5).

To complete the proof of (i), for each m we will construct an injection ι_m: $R_m \to R_{m+1}$, where $R_m := \{\alpha \in \Delta_\mathfrak{k}^+ \cap \Delta_m \mid (\lambda_m + \rho_m, \alpha) < 0\}$. Then $\iota := \iota_{k-1} \circ \cdots \circ \iota_s$: $R_s \to R_k$ will be an injection. Hence $\ell(E_s) = |R_s| \le |R_k| = \ell(E_k)$ where $|\cdot|$ denotes cardinality.

Let $I_{m+1} = \{c_1, \dots, c_{m+1}\}$ where $c_1 < \cdots < c_{m+1}$, and $I_m = \{c_1, \dots, c_{i'-1}, c_{i'+1}, \dots, c_{m+1}\}$ where $p_{q_0-1} + 1 \le c_{i'} \le p_{q_0}$. In view of the choice of Borel subalgebra $\mathfrak{b}_\mathfrak{k}$ we have $b_j = j$ for $1 \le j \le i'$, and $b_{i'} = i'$ is the largest element of $I_{m+1} \cap [p_{q_0-1} + 1, p_{q_0}]$. Set $\lambda_{m+1} + \rho_{m+1} =: u'_1\varepsilon_{c_1} + \cdots + u'_{m+1}\varepsilon_{c_{m+1}}$, $\lambda_m + \rho_m =: v_1\varepsilon_{c_1} + \cdots + v_{i'-1}\varepsilon_{c_{i'-1}} + v_{i'+1}\varepsilon_{i'+1} + \cdots + v_{m+1}\varepsilon_{c_{m+1}}$, and $u_i := u'_i - \frac{1}{2}$. Then, from (9.2), (9.3) and (9.4),

$$\begin{aligned} &u_i = v_i \quad \text{for} \quad 1 \le i \le p_{q-1}, \\ &u_{p_{q-1}+1} \ge v_{p_{q-1}+1} > \cdots > u_{i'-1} \ge v_{i'-1} > u_{i'}, \\ &u_i = v_i - 1 \quad \text{for} \quad i' < i \le m+1. \end{aligned} \tag{9.6}$$

Furthermore, the definition of I_m and the regularity of E_s and E_k, imply that $v_1, \dots, v_{p_{q-1}}$ are distinct and so are $v_{i'+1}, \dots, v_{m+1}$.

Let $\alpha \in R_m$. If $\alpha \in R_{m+1}$, we set $\iota_m(\alpha) := \alpha$. Assume that $\alpha = \varepsilon_{c_x} - \varepsilon_{c_y} \notin R_{m+1}$. Then $v_x - v_y < 0$ and $u_x - u_y \ge 0$. Now (9.6) implies $x < i'$ and $i' < y$. If $p_{q-1} < x$, we have $u_{i'} < v_x < v_y = u_y + 1$, and hence $(\lambda_{m+1} + \rho_{m+1}, \varepsilon_{c_{i'}} - \varepsilon_{c_y}) = u_{i'} - u_y < 0$.

In this case we set $\iota_m(\alpha) := \varepsilon_{c_{i'}} - \varepsilon_{c_y}$. If $x \leq p_{q-1}$, then $v_x = u_x = u_y = v_y - 1$. If there exists $p_{q-1} + 1 \leq z < i'$ with $v_z \leq u_y \leq u_z$, we have $(\lambda_{m+1} + \rho_{m+1}, \varepsilon_{c_x} - \varepsilon_{c_z}) = u_x - u_z \leq 0$. Notice that (9.5) rules out the equality $u_y = u_z$, and hence $(\lambda_{m+1}+\rho_{m+1}, \varepsilon_{c_x}-\varepsilon_{c_z}) = u_x - u_z < 0$. Furthermore, $(\lambda_m+\rho_m, \varepsilon_{c_x}-\varepsilon_{c_z}) = v_x - v_z \geq 0$, and thus $\varepsilon_{c_x} - \varepsilon_{c_z} \notin R_m$. In that case we set $\iota_m(\alpha) := \varepsilon_{c_x} - \varepsilon_{c_z}$. Finally, if there is no z as above, $(\lambda_{m+1} + \rho_{m+1}, \varepsilon_{c_x} - \varepsilon_{c_{i'}}) + (\lambda_{m+1} + \rho_{m+1}, \varepsilon_{c_{i'}} - \varepsilon_{c_y}) = u_x - u_y = 0$, and therefore at least one of the terms in the sum is nonpositive. On the other hand, (9.5) again rules out the vanishing of either term, and thus one of them is positive and the other one negative. In this last case we set $\iota_m(\alpha)$ to be the root ($\varepsilon_{c_x} - \varepsilon_{c_{i'}}$ or $\varepsilon_{c_{i'}} - \varepsilon_{c_y}$) corresponding to the negative term. The construction of ι_m ensures that it is an injection, and the proof of (i) is complete.

(ii) We claim that when $\ell(E_s) = \ell(E_k)$ and both E_s and E_k are regular, each E_m (as constructed in the proof of (i)) is also regular. Indeed, let $R'_m := \{\alpha \in \Delta^+_{\mathfrak{k}} \cap \Delta_m \mid (\lambda_m + \rho_m, \alpha) \leq 0\}$. Then there is an injection $\iota'_m\colon R'_m \to R'_{m+1}$ whose construction is almost identical to the one above. Notice that $R_m \subset R'_m$, and $R_s = R'_s$, $R_k = R'_k$. The assumption $\ell(E_s) = \ell(E_k)$ implies now that both $\iota\colon R_s \to R_k$ and $\iota' := \iota'_{k-1} \circ \cdots \circ \iota'_s\colon R'_s \to R'_k$ are bijections. Hence $|R'_m| = \ell(E_s) = \ell(E_k) = |R_m|$ for every $s \leq m \leq k$, and thus $R_m = R'_m$. This means that E_m is regular. $\square$

The following proposition is crucial in the proof of Theorem 11.1. Note the connection with the notions of cohomological finiteness in [20, Section 4].

PROPOSITION 9.7. *Let* $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ *be root-reductive,* $\mathfrak{p} = \varinjlim \mathfrak{p}_n$ *be a parabolic subalgebra of* $\mathfrak{g}$*, and* $E = \varinjlim E_n$ *be a locally irreducible locally finite* $\mathfrak{p}$*-module.*

(i) *Either* $\lim \ell(E_n) = \infty$*, or there exist* q_0 *and* m *such that* $\ell(E_n) = q_0$ *for every* $n \geq m$*.*

(ii) *Assume that* $\ell(E_n) = q_0$ *for* $n \geq m$ *and that* E_n *is regular for infinitely many* n*. Then there exist a Borel subalgebra* $\mathfrak{b} \subset \mathfrak{p}$*, an integer* $m' \geq m$*, and an element* $w \in \mathcal{W}$ *of length* q_0 *with respect to* $\mathfrak{b}$*, such that* $\mu_n := w(\lambda_n + \rho_n) - \rho_n$ *is a* $\mathfrak{b}_n$*-dominant weight for* $n \geq m'$*, where* $\mathfrak{b}_n := \mathfrak{b} \cap \mathfrak{g}_n$ *and* $\rho_n := \rho_{\mathfrak{b}_n}$*. Furthermore,* E_n *is regular for every* $n \geq m'$*.*

Proof. A direct verification shows that it is enough to prove the proposition under the assumption that $\mathfrak{g}$ is simple. More precisely, if $\mathfrak{g}^t$ for $t \in T$ are as in Proposition 4.1(iii), then $E_n = \bigotimes_{t \in T} E^t_n$, E^t_n being irreducible finite dimensional $\mathfrak{p}^t_n := \mathfrak{p} \cap \mathfrak{g}^t$-modules. Using the observation that $\ell(E_n) = \sum_{t \in T} \ell(E^t_n)$, one notes that the claims of the proposition follow directly from the same claims for all $\mathfrak{p}^t$-modules $E^t = \varinjlim E^t_n$. If $\mathfrak{g}^t$ is finite dimensional there is nothing to prove, so in the rest of the proof we will assume that $\mathfrak{g} = \mathfrak{a}(\infty)$, $\mathfrak{b}(\infty)$, $\mathfrak{c}(\infty)$, or $\mathfrak{d}(\infty)$.

We start by introducing notation. Recall decomposition (4.5) and the fact that $S_{\mathfrak{p}}$ is an ordered set. Consider the ordered set $\bar{S}_{\mathfrak{p}} := S_{\mathfrak{p}} \cup \{-\infty, \infty\}$, where $-\infty < s < \infty$ for every $s \in S_{\mathfrak{p}}$. Note that as an ordered set $\bar{S}_{\mathfrak{p}}$ is isomorphic to a subset of $\mathbb{R} \cup \{-\infty, \infty\}$. In the following, s_0 denotes the unique element of $S_{\mathfrak{p}}$ for which $\mathfrak{l}^{s_0} \ncong \mathfrak{gl}(p)$ or $\mathfrak{gl}(\infty)$; if such an element does not exist, we set

$s_0 := \infty \in \bar{S}_\mathfrak{p}$ and $\mathfrak{l}^{s_0} := 0$. For $s_1 \leq s_2 \in \bar{S}_\mathfrak{p}$, let $\Delta_n^{[s_1,s_2]}$ denote the subset of Δ_n consisting of all integral linear combinations of simple roots of $\mathfrak{b}_n$ whose nodes in the Dynkin diagram of $\mathfrak{g}_n$ correspond to elements $s \in S_{\mathfrak{p}_n}$ with $s_1 \leq s \leq s_2$. Put $\Delta^{[s_1,s_2]} := \bigcup_n \Delta_n^{[s_1,s_2]}$. Let $(\mathfrak{g}^{[s_1,s_2]})'$ be the Lie subalgebra of $\mathfrak{g}$ generated by all root spaces $\mathfrak{g}^\alpha$ for $\alpha \in \Delta^{[s_1,s_2]}$ (see §4) and let $\mathfrak{g}^{[s_1,s_2]} := (\mathfrak{g}^{[s_1,s_2]})' + (\bigoplus_{s_1 \leq s \leq s_2} \mathfrak{l}^s)$. When $s_1 = s_2 = s$ we often write simply $\mathfrak{g}^s$ instead of $\mathfrak{g}^{[s_1,s_2]}$. Set also $\mathfrak{p}^{[s_1,s_2]} := \mathfrak{p} \cap \mathfrak{g}^{[s_1,s_2]}$. Next, define the locally irreducible locally finite $\mathfrak{p}^{[s_1,s_2]}$-module $E^{[s_1,s_2]}$ as follows. Fix a Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{p}$, and let λ_n be the $\mathfrak{b}'_n$-highest weight of E_n. Let $E_n^{[s_1,s_2]}$ be the irreducible $\mathfrak{p}_n^{[s_1,s_2]}$-module with $\mathfrak{b}'_n \cap \mathfrak{p}_n^{[s_1,s_2]}$-highest weight $\lambda_{n_{|\mathfrak{h}_n^{[s_1,s_2]}}}$, where $\mathfrak{h}_n^{[s_1,s_2]} := \mathfrak{h} \cap \mathfrak{g}_n^{[s_1,s_2]}$. The modules $E_n^{[s_1,s_2]}$ form a direct system and we put $E^{[s_1,s_2]} := \varinjlim E_n^{[s_1,s_2]}$. We define similarly the Lie algebras $\mathfrak{g}^{(s_1,s_2)}$, $\mathfrak{g}^{(s_1,s_2]}$ and $\mathfrak{g}^{[s_1,s_2)}$, their respective parabolic subalgebras $\mathfrak{p}^{(s_1,s_2)}$, $\mathfrak{p}^{(s_1,s_2]}$ and $\mathfrak{p}^{[s_1,s_2)}$, and the corresponding modules $E^{(s_1,s_2)}$, $E^{(s_1,s_2]}$, and $E^{[s_1,s_2)}$. Throughout the proof, an upper index ${}^{[s_1,s_2]}$, ${}^{(s_1,s_2)}$, ${}^{[s_1,s_2)}$ or ${}^{(s_1,s_2]}$ always refers to a corresponding algebra, module or Weyl group. For example, $\mathfrak{h}_n^{(s_1,s_2)} = \mathfrak{h} \cap \mathfrak{g}_n^{(s_1,s_2)}$, $\mathcal{W}^{[s_1,s_2]}$ is the Weyl group of $\mathfrak{g}^{[s_1,s_2]}$, W^s is the Weyl group of $\mathfrak{g}^s = \mathfrak{g}^{[s,s]}$, etc. Finally, we set $R_n := \{\alpha \in \Delta_n^+ \mid (\lambda_n + \rho_n, \alpha) < 0\}$.

Here is the idea of the proof. Assume $\varinjlim \ell(E_n) \neq \infty$. For $\mathfrak{g} = \mathfrak{a}(\infty)$, (i) follows from Lemma 9.1(i). Applying Lemma 9.1(ii) to the $\mathfrak{p}_n^{[s',s'']}$-module $E_n^{[s',s'']}$ for various $s' \leq s''$, we are able locate the roots $\alpha \in R_n$ for n large enough. Namely, we show that there exist finitely many pairs $s'_i \leq s''_j$ such that $\mathfrak{g}^{(s'_i,s''_i)}$ is finite dimensional and every $\alpha \in R_n$ belongs to some $\Delta^{[s'_i,s''_i]}$. Therefore we can find finitely many finite dimensional subalgebras $\mathfrak{g}^i \cong \mathfrak{a}(p_i)$ (or $\mathfrak{gl}(p_i)$) of $\mathfrak{g}$ with $\mathfrak{g}^i \supset \mathfrak{g}^{(s'_i,s''_i)}$ so that $R_n \subset \bigcup \Delta_{\mathfrak{g}^i}$. Now we fix a Borel subalgebra $\bar{\mathfrak{b}}$ of $\bigoplus \mathfrak{g}^i$ and choose $\mathfrak{b}$ so that every simple root of $\bar{\mathfrak{b}}$ is a simple root of $\mathfrak{b}$. The desired element w exists and is of the claimed length because $\bigoplus \mathfrak{g}^i$ is finite dimensional. The regularity of all E_n now follows easily from Lemma 9.1(ii). For $\mathfrak{g} \neq \mathfrak{a}(\infty)$, we consider two cases depending on whether $\mathfrak{g}^{s_0}$ is finite or infinite dimensional. If $\mathfrak{g}^{s_0}$ is infinite dimensional, we first apply the proposition to $\mathfrak{g}^{(-\infty,s_0)} \cong \mathfrak{gl}(\infty)$, $\mathfrak{gl}(p)$, $\mathfrak{a}(\infty)$, or $\mathfrak{a}(p)$ to obtain $w_1 \in \mathcal{W}$. We then show that no sign changes are involved in the Weyl group elements that make $\lambda_n + \rho_n$ dominant, and find an element $w_2 \in \mathcal{W}$ which involves only reflections along roots of $\mathfrak{g}$ which are not roots of $\mathfrak{g}^{(-\infty,s_0)}$ or $\mathfrak{g}^{s_0}$. The desired element w then equals $w_2 \circ w_1$. If $\mathfrak{g}^{s_0}$ is finite dimensional, we first apply the proposition to a natural subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$, $\bar{\mathfrak{g}} \cong \mathfrak{a}(\infty)$ to obtain $w_1 \in \mathcal{W}$ which does not involve sign changes. The element w_1 maps the weights $\lambda_n + \rho_n$ into weights $\nu_n + \rho_n$, where ν_n are represented in coordinates by decreasing sequences of (not necessarily positive) integers or half-integers. We then find an element w_2 which maps $\nu_n + \rho_n$ into dominant weights. The element $w := w_2 \circ w_1 \in \mathcal{W}$ then satisfies the proposition. The details are presented separately for $\mathfrak{g} = \mathfrak{a}(\infty)$ and $\mathfrak{g} \neq \mathfrak{a}(\infty)$.

The case $\mathfrak{g} = \mathfrak{a}(\infty)$. Statement (i) follows immediately from Lemma 9.1(i). We now prove (ii). Recall that $\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j\}$ and $\Delta_n = \{\varepsilon_i - \varepsilon_j \mid$

$1 \leq i \neq j \leq n\}$. The parabolic subalgebra $\mathfrak{p}$ fixes a partition $\mathbb{N} = \sqcup_{s\in S_\mathfrak{p}} I^s$ such that $\Delta^{[s_1,s_2]} = \{\pm(\varepsilon_i - \varepsilon_j) \mid i \in I^{s_1'}, j \in I^{s_2'},$ where $s_1 \leq s_1' \leq s_2' \leq s_2\}$. The Borel subalgebra $\mathfrak{b}'$ fixes a linear order $<_{\mathfrak{b}'}$ on $\mathbb{N}$: $i <_{\mathfrak{b}'} j$ if $\varepsilon_i - \varepsilon_j \in \Delta^+$. Notice also that each of the Lie algebras $\mathfrak{g}^{[s_1,s_2]}$, $\mathfrak{g}^{(s_1,s_2]}$, $\mathfrak{g}^{[s_1,s_2)}$ and $\mathfrak{g}^{(s_1,s_2)}$ is isomorphic to $\mathfrak{gl}(\infty)$, $\mathfrak{gl}(p)$, $\mathfrak{a}(\infty)$, or $\mathfrak{a}(p)$ for some p. Fix an infinite sequence $n_1 < n_2 < \cdots$, $n_i \in \mathbb{N}$, such that E_{n_k} is a regular module of length q_0 for every k. Lemma 9.1(ii) implies the existence of root subalgebras $\mathfrak{g}_n' \cong \mathfrak{a}(n-1)$ of $\mathfrak{g}$ and regular $\mathfrak{g}_n'$-modules E_n' such that $\mathfrak{g}_{n_k}' = \mathfrak{g}_{n_k}$ and $E_{n_k}' = E_{n_k}$ for $n, n_k \geq m'' := n_1$. Denote by λ_n' the $\mathfrak{b}_n'$-highest weight of E_n' and set $\lambda_n' := (\lambda_n')^1\varepsilon_1 + \cdots + (\lambda_n')^n\varepsilon_n$.

First we show that $\operatorname{rank}\mathfrak{g}^{(s',s'')} < q_0$ whenever $(\lambda_{m''}' + \rho_{m''}, \alpha) < 0$ for $\alpha = \varepsilon_i - \varepsilon_j \in \Delta_{m''}^+$ with $i \in I^{s'}, j \in I^{s''}$. For $n \geq m''$, let i_n be the $<_{\mathfrak{b}'}$-maximal element of $I^{s'} \cap \{1, \ldots, n\}$ and j_n be the $<_{\mathfrak{b}'}$-minimal element of $I^{s''} \cap \{1, \ldots, n\}$. We prove by induction on n that $(\lambda_n' + \rho_n, \alpha_n) < 0$ for $\alpha_n := \varepsilon_{i_n} - \varepsilon_{j_n}$. When $n = m''$, $(\rho_{m''}, \alpha) \geq (\rho_{m''}, \alpha_{m''})$ and $(\lambda_{m''}', \alpha) \geq (\lambda_{m''}', \alpha_{m''})$, thus $(\lambda_{m''}' + \rho_{m''}, \alpha_{m''}) \leq (\lambda_{m''}' + \rho_{m''}, \alpha) < 0$. Assume $(\lambda_n' + \rho_n, \alpha_n) < 0$. If $n+1 \notin I^s$ for any $s' < s < s''$, then $(\rho_{n+1}, \alpha_{n+1}) = (\rho_n, \alpha_n)$, and by the branching rule $(\lambda_{n+1}')^{i_{n+1}} \leq (\lambda_n')^{i_n}$, $(\lambda_{n+1}')^{j_{n+1}} \geq (\lambda_n')^{j_n}$. Hence $(\lambda_{n+1}', \alpha_{n+1}) \leq (\lambda_n', \alpha_n)$, which implies $(\lambda_{n+1}' + \rho_{n+1}, \alpha_{n+1}) \leq (\lambda_n' + \rho_n, \alpha_n) < 0$. If $n+1 \in I^s$ for some $s' < s < s''$, then $(\rho_{n+1}, \alpha_{n+1}) = (\rho_n, \alpha_n) + 1$. Furthermore, $(\lambda_{n+1}')^{i_{n+1}} = (\lambda_n')^{i_n}$, $(\lambda_{n+1}')^{j_{n+1}} = (\lambda_n')^{j_n}$, and $(\lambda_{n+1}', \alpha_{n+1}) = (\lambda_n', \alpha_n)$. Thus $(\lambda_{n+1}' + \rho_{n+1}, \alpha_{n+1}) = (\lambda_n' + \rho_n, \alpha_n) + 1 \leq 0$. But since E_n' is regular, $(\lambda_{n+1}' + \rho_{n+1}, \alpha_{n+1}) \neq 0$ and then $(\lambda_{n+1}' + \rho_{n+1}, \alpha_{n+1}) < 0$. The inequality $(\lambda_n' + \rho_n, \alpha_n) < 0$ implies now $\operatorname{rank}\mathfrak{g}^{(s',s'')} < q_0$ as the height of α_n equals $\operatorname{rank}(\mathfrak{g}_n')^{(s',s'')} + 1$.

We show next by induction on q_0, that there exists a sequence $s_1' \leq s_1'' \leq s_2' \leq s_2'' \leq \cdots \leq s_r' \leq s_r'' \in \bar{S}_\mathfrak{p}$ such that, when $n \geq m''$, $\ell((E_n')^{[s_1',s_1'']}) + \cdots + \ell((E_n')^{[s_r',s_r'']}) = q_0$ and $\operatorname{rk}(\mathfrak{g}_n')^{(s_i',s_i'')} \leq q_0$ for every $1 \leq i \leq r$. If $q_0 = 0$ we set $r := 0$. Assume $q_0 > 0$. Let, as above, $(\lambda_{m''}' + \rho_{m''}, \alpha) < 0$ for $\alpha = \varepsilon_i - \varepsilon_j \in \Delta_{m''}^+$ with $i \in I^{s'}, j \in I^{s''}$ and $s' < s''$. Then $\ell((E_n')^{(-\infty,s']}) + \ell((E_n')^{[s',s'']}) + \ell((E_n')^{[s'',\infty)}) \leq q_0$ for every $n \geq m''$. If $\ell((E_n')^{(-\infty,s']}) + \ell((E_n')^{[s',s'']}) + \ell((E_n')^{[s'',\infty)}) = q_0$, we apply the induction assumption to the $\mathfrak{p}^{(-\infty,s']}$-module $E^{(-\infty,s']}$ to get $s_1', s_1'', \ldots, s_{r'-1}', s_{r'-1}''$, and to the $\mathfrak{p}^{[s'',\infty)}$-module $E^{[s'',\infty)}$ to get $s_{r'+1}', s_{r'+1}'', \ldots, s_r', s_r''$. By setting $s_{r'}' := s', s_{r'}'' := s''$, we obtain a sequence with the desired properties. If $\ell((E_n')^{(-\infty,s']}) + \ell((E_n')^{[s',s'']}) + \ell((E_n')^{[s'',\infty)}) < q_0$, there exists $\beta = \varepsilon_{i'} - \varepsilon_{j'} \in \Delta_{m''}^+$ where $i' \in I^{\bar{s}'}$, $j' \in I^{\bar{s}''}$ with $(\lambda_{m''}' + \rho_{m''}, \beta) < 0$ and $\ell((E_n')^{[\bar{s}',\bar{s}'']}) > \ell((E_n')^{[s',s'']})$. Then $\ell((E_n')^{(-\infty,\bar{s}']}) + \ell((E_n')^{[\bar{s}',\bar{s}'']}) + \ell((E_n')^{[\bar{s}'',\infty)}) \leq q_0$ and we again consider two cases. For $\ell((E_n')^{(-\infty,\bar{s}']}) + \ell((E_n')^{[\bar{s}',\bar{s}'']}) + \ell((E_n')^{[\bar{s}'',\infty)}) = q_0$, the desired sequence is constructed as above by replacing s' and s'' by $\bar{s}'$ and $\bar{s}''$ respectively. If $\ell((E_n')^{(-\infty,\bar{s}']}) + \ell((E_n')^{[\bar{s}',\bar{s}'']}) + \ell((E_n')^{[\bar{s}'',\infty)}) < q_0$, we replace in a similar manner $\bar{s}'$ and $\bar{s}''$ by $\bar{\bar{s}}'$ and $\bar{\bar{s}}''$ respectively. The process is finite because $q_0 \geq \ell((E_n')^{[\bar{\bar{s}}',\bar{\bar{s}}'']}) > l((E_n')^{[\bar{s}',\bar{s}'']}) > \ell((E_n')^{[s',s'']})$.

Fix a sequence $s_1' \leq s_1'' \leq s_2' \leq s_2'' \leq \cdots \leq s_r' \leq s_r'' \in \bar{S}_\mathfrak{p}$ as above. Let $m' = n_{k_0} \geq m$ be such that $\mathfrak{g}_{m'}^{(s_i',s_i'')} = \mathfrak{g}^{(s_i',s_i'')}$ for $1 \leq i \leq r$, $\operatorname{rk}\mathfrak{g}_{m'}^{s_i'} = \min\{\operatorname{rk}\mathfrak{g}^{s_i'}, 2q_0\}$ and $\operatorname{rk}\mathfrak{g}_{m'}^{s_i''} = \min\{\operatorname{rk}\mathfrak{g}^{s_i''}, 2q_0\}$. Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{p}$ with the property that whenever $\operatorname{rk}\mathfrak{g}^{s'_i} > 2q_0$ (respectively, $\operatorname{rk}\mathfrak{g}^{s''_i} > 2q_0$), there are q_0 elements $x^{q_0}_{s'_i} >_{\mathfrak{b}} \cdots >_{\mathfrak{b}} x^1_{s'_i} \in I^{s'_i}$ (resp. $y^1_{s''_i} <_{\mathfrak{b}} \cdots <_{\mathfrak{b}} y^{q_0}_{s'_i} \in I^{s''_i}$) such that $x >_{\mathfrak{b}} x^{q_0}_{s'_i}$ (resp. $y^{q_0}_{s''_i} <_{\mathfrak{b}} y$) for any other $x \in I^{s'_i}$ (resp. $y \in I^{s''_i}$). Let $\lambda_n = \lambda^1_n\varepsilon_1 + \cdots + \lambda^n_n\varepsilon_n$ be the $\mathfrak{b}_n$-highest weight of E_n.

Fix $n \geq m'$. If $\alpha \in R_{m'}$, then $\alpha = \varepsilon_x - \varepsilon_y$ for $x \in I^{s_x}$, $y \in I^{s_y}$. The construction of the sequence $s'_1, s''_1, \ldots, s'_r, s''_r$ and the definition of $\mathfrak{b}$ implies $s'_i \leq s_x < s_y \leq s''_i$ for some i, $1 \leq i \leq r$. If $s_x = s'_i$, there are at most q_0 elements of $I^{s'_i}$ $<_{\mathfrak{b}}$-greater than x, and if $s_y = s''_i$, there are at most q_0 elements of $I^{s''_i}$ $<_{\mathfrak{b}}$-smaller than y. Furthermore, the choice of $\mathfrak{b}$ gives $(\rho_n, \alpha) = (\rho_{m'}, \alpha)$ and the branching rule implies $\lambda^x_n \leq \lambda^x_{m'}$ and $\lambda^y_n \geq \lambda^y_{m'}$. Hence $(\lambda_n, \alpha) = \lambda^x_n - \lambda^y_n \leq \lambda^x_{m'} - \lambda^y_{m'} = (\lambda_{m'}, \alpha) < 0$, and thus $\alpha \in R_n$. This establishes the inclusion $R_{m'} \subset R_n$. In particular $R_{m'} = R_{n_k}$ whenever $n_k \geq m'$. If, on the other hand, $\beta \in \Delta^+_n \backslash R_n$, fix k so that $n_k \geq n$. If $\beta = \varepsilon_x - \varepsilon_y$, with $x \in I^{s_x}$, $y \in I^{s_y}$, assume that x is the ${j_x}^{th}$ largest element (with respect to $<_{\mathfrak{b}}$) of $I^{s_x} \cap \{1, \ldots, n\}$, and y is the ${j_y}^{th}$ smallest element (with respect to $<_{\mathfrak{b}}$) of $I^{s_y} \cap \{1, \ldots, n\}$. Let x' be the ${j_x}^{th}$ largest element of $I^{s_x} \cap \{1, \ldots, n_k\}$, and let y' be the ${j_y}^{th}$ smallest element of $I^{s_y} \cap \{1, \ldots, n_k\}$. Set $\beta' := \varepsilon_{x'} - \varepsilon_{y'}$. The assumption $\beta' \in R_{n_k}$ would give $\beta = \beta'$, and hence $\beta \in R_n$, which contradicts the choice of β. Therefore $(\lambda_{n_k} + \rho_{n_k}, \beta') > 0$. Notice that $(\rho_n, \beta) = (\rho_{n_k}, \beta')$, and, by the branching rule, $\lambda^x_n \geq \lambda^{x'}_{n_k}$, $\lambda^y_n \leq \lambda^{y'}_{n_k}$. Thus $(\lambda_n, \beta) = \lambda^x_n - \lambda^y_n \geq \lambda^{x'}_{n_k} - \lambda^{y'}_{n_k} = (\lambda_{n_k}, \beta')$. Now it is clear that $(\lambda_n + \rho_n, \beta) \geq (\lambda_{n_k} + \rho_{n_k}, \beta') > 0$. This proves that E_n is regular. Hence $|R_n| = |R_{m'}|$ which yields $R_n = R_{m'}$. The latter combined with the observation that every $\alpha \in R_{m'}$ has the same height in Δ_n for any $n \geq m'$ implies also the existence of $w \in \mathcal{W}$ as required in (ii).

The case $\mathfrak{g} \neq \mathfrak{a}(\infty)$. The cases $\mathfrak{g} = \mathfrak{b}(\infty)$, $\mathfrak{c}(\infty)$ and $\mathfrak{d}(\infty)$ are treated in an almost identical way. We present the details for $\mathfrak{g} = \mathfrak{b}(\infty)$ only.

In the rest of the proof $\mathfrak{g} = \mathfrak{b}(\infty)$. We establish (i) and (ii) simultaneously. If there are only finitely many regular modules among E_n, (i) follows from the definition of $\ell(E_n)$ (as $E_n \prec E_{n+1}$ and hence $\ell(E_n) \leq \ell(E_{n+1})$ whenever E_{n+1} is not regular) and (ii) is an empty statement. Assume that infinitely many E_n are regular and $\lim \ell(E_n) \neq \infty$. Then there exists q_0 and an infinite sequence $n_1 < n_2 < \cdots$, $n_k \in \mathbb{N}$, such that E_{n_k} is a regular module of length q_0 for every k. Indeed, $\lim \ell(E_n) \neq \infty$ implies the existence of q'_0 and a sequence $\{n'_k\}$ for which $\ell(E_{n'_k}) = q'_0$. If there are infinitely many regular modules among $E_{n'_k}$, set $q_0 := q'_0$ and take $\{n_k\}$ to be the subsequence of $\{n'_k\}$ corresponding to regular modules E_{n_k}. If there are only finitely many regular modules among $E_{n'_k}$, then $\ell(E_n) \leq q'_0$ for every n, as there is $n'_k > n$ with $E_{n'_k}$ not regular, and hence $\ell(E_n) \leq \ell(E_{n'_k}) = q'_0$ by the definition of $\ell(E_{n'_k})$. In particular, $\ell(E_n) \leq q'_0$ for every regular E_n, and thus, for some $q_0 \leq q'_0$, $\ell(E_n) = q_0$ for infinitely many regular E_n. Fix q_0 and a sequence $n_1 < n_2 < \cdots$ so that E_{n_k} is a regular module of length q_0 for every k.

Recall that $\Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j\}$ and $\Delta_n = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. The Borel subalgebra $\mathfrak{b}'$ determines the sign function

$$\operatorname{sgn}\colon \mathbb{N} \to \{\pm 1\}, \quad \operatorname{sgn}(i) = 1 \text{ if and only if } \varepsilon_i \in \Delta^+$$

and a linear order $<_{\mathfrak{b}'}$ on $\mathbb{N}$: $i <_{\mathfrak{b}'} j$ if $\operatorname{sgn}(i)\varepsilon_i - \operatorname{sgn}(j)\varepsilon_j \in \Delta^+$. The parabolic subalgebra $\mathfrak{p}$ determines a partition $\mathbb{N} = \sqcup_{s\in S_{\mathfrak{p}}} I^s$ such that $\Delta^{[s_1,s_2]} = \{\pm(\operatorname{sgn}(i)\varepsilon_i - \operatorname{sgn}(j)\varepsilon_j) \mid i \in I^{s'_1}, j \in I^{s'_2},$ where $s_1 \leq s'_1 \leq s'_2 \leq s_2\}$ and $\Delta^{s_0} = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j \in I^{s_0}\}$.

Assume first that $\mathfrak{g}^{s_0}$ is infinite dimensional and let $\operatorname{rk}\mathfrak{g}^{s_0}_{m_0} > q_0$ for some m_0. Then $(\lambda_{n_k} + \rho_{n_k}, \operatorname{sgn}(i)\varepsilon_i) > 0$ for $1 \leq i \leq n_k$ and $n_k \geq m_0$. Indeed, if $i \in I^{s_0}$, $(\lambda_{n_k}, \operatorname{sgn}(i)\varepsilon_i) \geq 0$ because $\lambda_{n_k|\mathfrak{h}^{s_0}_{n_k}}$ is a $\mathfrak{b}'^{s_0}_{n_k}$-dominant weight, and $(\rho_{n_k}, \operatorname{sgn}(i)\varepsilon_i) > 0$. If, on the other hand, $i \notin I^{s_0}$, then $\operatorname{sgn}(i)\varepsilon_i \in \Delta_{n_k}$ is of height greater than q_0, and $(\lambda_{n_k} + \rho_{n_k}, \operatorname{sgn}(i)\varepsilon_i) < 0$ would imply $\ell(E_{n_k}) > q_0$. Notice that $\ell(E_n^{(-\infty,s_0)}) \leq \ell(E_n)$, and hence the proposition applied to $\mathfrak{p}^{(-\infty,s_0)} \subset \mathfrak{g}^{(-\infty,s_0)}$ and the $\mathfrak{p}^{(-\infty,s_0)}$-module $E^{(-\infty,s_0)}$ yields integers $m'_1 \geq m_0$ and q_1, an element $w_1 \in \mathcal{W}^{(-\infty,s_0)}$, and a Borel subalgebra $\tilde{\mathfrak{b}}'$ of $\mathfrak{p}^{(-\infty,s_0)}$. Let $\mathfrak{b}''$ be a Borel subalgebra of $\mathfrak{p}$ for which $\mathfrak{b}'' \cap \mathfrak{p}^{(-\infty,s_0)} = \tilde{\mathfrak{b}}'$ and $\mathfrak{b}'' \cap \mathfrak{p}^{s_0} = \mathfrak{b}' \cap \mathfrak{p}^{s_0}$. Since $\mathcal{W}^{(-\infty,s_0)} \subset \mathcal{W}$, w_1 has length q_1 respect to $\mathfrak{b}''$. Set $\nu_n := w(\lambda_n + \rho_n) - \rho_n$ and let ${}^{w_1}E_n$ be the irreducible ${}^{w_1}\mathfrak{p}_n$-module with $\mathfrak{b}''_n$-highest weight ν_n. We claim that there exists a ${}^{w_1}\mathfrak{p}$-module of the form ${}^{w_1}E := \varinjlim_{n \geq m'_1} {}^{w_1}E_n$. Indeed, the existence of ${}^{w_1}E$ follows from Theorem 11.1(iii) below for $G = A(\infty)$ and $E^{(\infty,s_0)}$, see the discussion after that Theorem. (As the proof of Theorem 11.1 for $G = A(\infty)$ does not depend on the case $G = B(\infty)$, $C(\infty)$ or $D(\infty)$, there is no contradiction in applying Theorem 11.1(iii) here.) Clearly ${}^{w_1}E_{n_k}$ is regular of length $q_0 - q_1$ for every $n_k \geq m'_1$.

Notice that $\mathfrak{b}'$ may be fixed from the very beginning so that there are at least $q_0 - q_1$ $<_{\mathfrak{b}'\cap\mathfrak{p}^{s_0}}$-smallest elements in I^{s_0}. Since $\mathfrak{b}'' \cap \mathfrak{p}^{s_0} = \mathfrak{b}' \cap \mathfrak{p}^{s_0}$, $\mathfrak{b}''$ has the same property. Let $y^1 <_{\mathfrak{b}''} \cdots <_{\mathfrak{b}''} y^{q_0-q_1}$ be the $q_0 - q_1$ $<_{\mathfrak{b}''}$-smallest elements of I^{s_0}. Observe that, for $n_k \geq m'_1$, $(\nu_{n_k} + \rho_{\mathfrak{b}''_{n_k}}, \alpha) < 0$ with $\alpha \in \Delta_{\mathfrak{b}''}$ implies that $\alpha = \operatorname{sgn}(x)\varepsilon_x - \operatorname{sgn}(y^i)\varepsilon_{y^i}$ for some $x \notin I^{s_0}$ and $1 \leq i \leq q_0$. Denote by $l_{k,i}$ the number of indices x for which $(\nu_{n_k} + \rho_{\mathfrak{b}''_{n_k}}, \operatorname{sgn}(x)\varepsilon_x - \operatorname{sgn}(y^i)\varepsilon_{y^i}) < 0$. The definition of the length of ${}^{w_1}E_{n_k}$ implies $\sum_{1\leq i\leq q_0-q_1} l_{k,i} = q_0 - q_1$, and dominance of $\nu_{n_k|\mathfrak{h}^{s_0}_{n_k}}$ gives $l_{k,1} \geq l_{k,2} \geq \cdots \geq l_{k,q_0-q_1}$. On the other hand, the branching rule implies $l_{k',i} \geq l_{k,i}$ for $k' \geq k$. Combining these facts about l_{k_i}, we obtain the existence of $l_1, \ldots, l_{q_0-q_1}$ with $l_{k,i} = l_i$ whenever $n_k \geq m'_1$. It is now easy to see that there is a Borel subalgebra $\mathfrak{b}$ of ${}^{w_1}\mathfrak{p}$ for which there are l_1 $<_{\mathfrak{b}}$-largest elements of $\mathbb{N}\backslash I^{s_0}$ and $\mathfrak{b} \cap \mathfrak{g}^{s_0} = \mathfrak{b}'' \cap \mathfrak{g}^{s_0}$. Fix such a $\mathfrak{b}$ and denote by $x^1 >_{\mathfrak{b}} \cdots >_{\mathfrak{b}} x^{l_1}$ the l_1 $<_{\mathfrak{b}}$-largest elements of $\mathbb{N}\backslash I^{s_0}$. Let $m' = n_{k_0} \geq m'_1$ be larger than $x^1, \ldots, x^{l_1}$. For $n_k \geq m'$ the element $w_{n_k} \in \mathcal{W}_{n_k}$ which makes $\nu_{n_k} + \rho_{\mathfrak{b}_{n_k}}$ dominant permutes only $x_1, \ldots, x_{l_1}, y_1, \ldots, y_{q_0-q_1}$. Furthermore, the equality $l_{k,i} = l_i$ implies that w_{n_k} does not depend on k. Set $w_2 := w_{n_k}$. Finally, one observes that each E_n is regular, as otherwise $(\nu_n + \rho_{\mathfrak{b}_n}, \alpha) = 0$ would imply $(\nu_{n_k} + \rho_{\mathfrak{b}_{n_k}}, \alpha) < 0$ for $n_k < n$ and $(\nu_{n_k} + \rho_{\mathfrak{b}_{n_k}}, \alpha) > 0$ for $n_k > n$, i.e., $\ell({}^{w_1}E_{n_k}) > \ell({}^{w_1}E_{n_{k_0}})$ for $n_k > n_0$. Hence $\mu_n = w(\lambda_n + \rho_n) - \rho_n$ is a regular $\mathfrak{b}_n$-dominant weight for $w := w_2 \circ w_1$. This completes the proof when $\mathfrak{g}^{s_0}$ is infinite dimensional.

Assume that $\mathfrak{g}^{s_0}$ is finite dimensional. Since the details in this case are similar to the details in the previous case, we will only outline the argument. Fix m_0 with $\mathfrak{g}^{s_0}_{m_0} = \mathfrak{g}^{s_0}$. Set $\bar{\Delta} := \{\operatorname{sgn}(i)\varepsilon_i - \operatorname{sgn}(j)\varepsilon_j \mid i \neq j\}$. Let $\bar{\mathfrak{g}}$ denote the root subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}^\alpha$ for $\alpha \in \bar{\Delta}$. Note that $\bar{\mathfrak{g}} \cong \mathfrak{a}(\infty)$. If $\bar{\mathfrak{p}} := \mathfrak{p} \cap \bar{\mathfrak{g}}$ and $\bar{\mathfrak{b}}' := \mathfrak{b}' \cap \bar{\mathfrak{g}}$, then $\bar{\mathfrak{p}} \subset \bar{\mathfrak{g}}$ is a parabolic subalgebra and $\bar{\mathfrak{b}}' \subset \bar{\mathfrak{p}}$ is a Borel subalgebra. Let $\bar{E}_n$ be the irreducible $\bar{\mathfrak{p}}_n$-module with $\bar{\mathfrak{b}}'_n$-highest weight λ_n. For $n \geq m_0$, $\lambda_{n|\mathfrak{h}_n^{s_0}} = \lambda_{m_0|\mathfrak{h}_{m_0}^{s_0}}$ and thus $\bar{E} := \varinjlim \bar{E}_n$ is a well-defined $\bar{\mathfrak{p}}$-module where the injections $\bar{E}_n \hookrightarrow \bar{E}_{n+1}$ are the restrictions of the injections $E_n \hookrightarrow E_{n+1}$. Since $\ell(\bar{E}_{n_k}) \leq \ell(E_{n_k})$, the proposition applied to the $\bar{\mathfrak{p}}$-module $\bar{E}$ yields integers m_1 and q_1, an element $w_1 \in \bar{\mathcal{W}} \subset \mathcal{W}$, and a Borel subalgebra $\bar{\mathfrak{b}}''$ of $\bar{\mathfrak{p}}$. Let $\mathfrak{b}''$ be the Borel subalgebra of $\mathfrak{p}$ with roots $\Delta_{\bar{\mathfrak{b}}''} \cup \{\operatorname{sgn}(i)\varepsilon_i, \operatorname{sgn}(i)\varepsilon_i + \operatorname{sgn}(j)\varepsilon_j \mid i \neq j\}$. Exactly as in the case when $\mathfrak{g}^{s_0}$ is infinite dimensional, there exists a well-defined ${}^{w_1}\bar{\mathfrak{p}}$-module ${}^{w_1}\bar{E} := \varinjlim {}^{w_1}\bar{E}_n$. Set $\nu_n := w_1(\lambda_n + \rho_n) - \rho_n$. The weights ν_n are $\bar{\mathfrak{b}}''$-dominant but not necessarily $\mathfrak{b}''$-dominant: for $n_k \geq m_1$, at most $q_0 - q_1$ of the integers $(\nu_n + \rho_n, \operatorname{sgn}(i)\varepsilon_i)$ are negative. On the other hand, the branching rule for the modules ${}^{w_1}\bar{E}_n$ implies that if l_n is the number of integers i for which $(\nu_n + \rho_n, \operatorname{sgn}(i)\varepsilon_i) < 0$, then $\{l_n\}_{n \geq m_1}$ is a nondecreasing bounded sequence. Fix $m_2 \geq m_1$ such that $l_n = l_{m_2}$ for $n \geq m_2$.

As in the case when $\mathfrak{g}^{s_0}$ is infinite dimensional, there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{p}$ such that the order $<_{\bar{\mathfrak{b}}}$ admits q' largest elements $x^1 >_{\bar{\mathfrak{b}}} \cdots >_{\bar{\mathfrak{b}}} x^{q'}$. Let $m' := n_{k_0} \geq m_2$ be larger than $x^1, \ldots, x^{q'}$. If $\mathfrak{g}^{q'}$ is the Lie algebra generated by $\mathfrak{g}^\alpha$ for $\alpha \in \{\pm\varepsilon_{x^i}, \pm\varepsilon_{x^i} \pm \varepsilon_{x^j} \mid 1 \leq i \neq j \leq q'\}$, then, for $n_k \geq m'$, the element w_{n_k} that makes $\nu_{n_k} + \rho_{n_k}$ dominant belongs to $\mathcal{W}^{q'}$. We then notice that w_{n_k} does not depend on k, and set $w_2 := w_{n_k}$. Furthermore, by comparing E_n with both $E_{n_{k'}}$ and $E_{n_{k''}}$ for $m' \leq n_{k'} \leq n \leq n_{k''}$, we conclude that E_n is regular for $n \geq m'$. Finally, $\mu_n = w(\lambda_n + \rho_n) - \rho_n$ is a regular $\mathfrak{b}_n$-dominant weight for $w := w_2 \circ w_1$. The proof is complete. □

Part III. Cohomology.

10. Homogeneous spaces and G-sheaves. Let P be a parabolic subgroup of a locally reductive ind-group G. For each n the map ψ_n: $X_n = G_n/P_n \hookrightarrow G_{n+1}/P_{n+1} = X_{n+1}$ is a closed immersion of smooth proper varieties, and the union is the proper ind-variety

$$X = G/P := \bigcup_n G_n/P_n = \bigcup_n X_n.$$

The translation action $\tau\colon G\times X \to X$ is an ind-variety morphism. In other words, it endows X with the structure of a G-ind-variety.

Example 10.1.

(1) Let V be a vector space of countably infinite dimension and let $k \in \mathbb{N}$. Choose an ordered basis in V and use it to identify V with the natural representation of $G = A(\infty)$. Let P be the stabilizer in G of the span of the first k vectors in the basis of V. Then P is a maximal parabolic subgroup of G and the G-ind-variety G/P is identified with the Grassmannian $Gr(k, V)$ introduced in Example 1.2.

(2) Let $G = GL(2^\infty)$ and $P = \varinjlim P_n$, where P_n is the parabolic subgroup of $GL(2^n)$ containing the diagonal matrices and such that the roots of its Lie subalgebra $\mathfrak{p}_n$ are

$$\{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le 2^n\} \cup \{\varepsilon_{2s+1} - \varepsilon_{2s} \mid 1 \le s \le 2^{n-2}\}.$$

Then G_n/P_n is the flag space $Fl(1, 3, \ldots, 2^n - 1, \mathbb{C}^{2^n})$ of maximal flags of odd dimensional subspaces in $\mathbb{C}^{2^n}$. The ind-variety G/P is the union $\bigcup_n G_n/P_n$ where the closed immersions $\sigma_n\colon G_n/P_n \hookrightarrow G_{n+1}/P_{n+1}$ are described as follows. According to the construction of G, the natural representation $\mathbb{C}^{2^{n+1}}$ of G_{n+1} has a canonical G_n-module decomposition as $\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}$. The map

$$\sigma_n\colon Fl(1, 3, \ldots, 2^n - 1, \mathbb{C}^{2^n}) \hookrightarrow Fl(1, 3, \ldots, 2^{n+1} - 1, \mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n})$$

sends a flag $F_1 \subset F_3 \subset \cdots \subset F_{2^n-1}$ in $\mathbb{C}^{2^n}$ to the flag $F_1 \oplus \{0\} \subset F_3 \oplus \{0\} \subset \cdots \subset F_{2^n-1} \oplus \{0\} \subset \mathbb{C}^{2^n} \oplus F_1 \subset \mathbb{C}^{2^n} \oplus F_3 \subset \cdots \subset \mathbb{C}^{2^n} \oplus F_{2^n-1}$ in $\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}$.

(3) Let $G = PGL^{\mathrm{Ad}}(k^{2^\infty})$ and B be any Borel ind-subgroup as described in Section 5. Then $G/B = \bigcup_n G_n/B_n$, where for each n G_n/B_n is the space $Fl(1, 2, \ldots, k^{2^n} - 1, \mathbb{C}^{k^{2^n}})$ of maximal flags in $\mathbb{C}^{k^{2^n}}$. The G_n-equivariant closed immersion

$$\theta_n\colon Fl(1, 2, \ldots, k^{2^n} - 1, \mathbb{C}^{k^{2^n}}) \hookrightarrow Fl(1, 2, \ldots, k^{2^{n+1}} - 1, \mathbb{C}^{k^{2^{n+1}}})$$

is characterized by the property that the maximal flag in $\mathbb{C}^{k^{2^n}}$ stabilized by the pre-image of B_n in $GL(k^{2^n})$ is sent to the $pr_n^{-1}(H_{n+1})$-invariant maximal flag in $\mathbb{C}^{k^{2^{n+1}}}$ whose stabilizer in $GL(k^{2^{n+1}})$ is $pr_{n+1}^{-1}(B_{n+1})$, see Section 5.

The structure sheaf $\mathcal{O}_X$ of X is the inverse limit sheaf $\varprojlim \mathcal{O}_{X_n}$ of the structure sheaves $\mathcal{O}_{X_n}$ of X_n. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, or an $\mathcal{O}_X$-module for short, we define $\mathcal{F}$ to be a *G-homogeneous $\mathcal{O}_X$-module* (= *G-linearized $\mathcal{O}_X$-module*), or simply a *G-sheaf*, if for every $g \in G$ an $\mathcal{O}_X$-module isomorphism φ_g is given, $\varphi_g\colon \tilde{g}^*(\mathcal{F}) \to \mathcal{F}$, so that $\varphi_{g'} \circ \varphi_{g''} = \varphi_{g' \circ g''}$ for all $g', g'' \in G$. Here $\tilde{g}\colon X \to X$ stands for the composition $X \simeq g \times X \xrightarrow{\tau|_{g \times X}} X$. If $\mathcal{F}$ is G-sheaf, then its total cohomology $H^{\cdot}(X;\mathcal{F}) := \bigoplus_{n \geq 0} H^n(X;\mathcal{F})$ is a graded G-module, i.e., for each n $H^n(X,\mathcal{F})$ is endowed with a G-module structure. The structure map $G \times H^n(X;\mathcal{F}) \to H^n(X;\mathcal{F})$ is the map $(g,f) \mapsto \varphi_g^{H^n}(f)$, where $\varphi_g^{H^n}\colon H^n(X;\mathcal{F}) \to H^n(X;\mathcal{F})$ is the linear automorphism of $H^n(X;\mathcal{F})$ induced by φ_g.

If $E = \varinjlim E_n$ is a rational P-module, there are certain natural G-sheaves on X associated with E. Note first that, for each n, we have a closed G_n-equivariant immersion

$$\mathbb{E}_n := (G_n \times_{P_n} E_n) \hookrightarrow (G_{n+1} \times_{P_{n+1}} E_{n+1}) =: \mathbb{E}_{n+1}$$

of the homogeneous vector bundles of finite rank associated respectively with E_n and E_{n+1}. These immersions determine a G-ind-variety $\mathbb{E} := \varinjlim \mathbb{E}_n$, the G-homogeneous vector bundle over X associated to E. We define $\mathcal{O}_X(E)$ to be the sheaf of local sections of $\mathbb{E} \to X$. It is a locally free G-sheaf of $\mathcal{O}_X$-modules, and rank $\mathcal{O}_X(E) = \dim E$. In particular, $\dim E$ can be infinite and, when P/U_P has no noncommutative finite dimensional factors, we have either $\dim E = 1$ or $\dim E = \infty$.

Another natural G-sheaf associated with E is $\mathcal{O}_X(E^*)$. As noted earlier, E^* is a pro-rational P-module, and there is a well-defined inverse system

$$\cdots \to \mathcal{O}_{X_{n+1}}(E^*_{n+1}) \to \mathcal{O}_{X_n}(E^*_n) \to \cdots \to 0,$$

$\mathcal{O}_{X_n}(E^*_n)$ being the sheaf of local sections of $\mathbb{E}_n$. The sheaf $\mathcal{O}_X(E^*)$ is defined now as the inverse limit $\varprojlim \mathcal{O}_{X_n}(E^*_n)$ with the $\mathcal{O}_{X_n}(E^*_n)$ viewed as sheaves of $\mathcal{O}_X$-modules supported on X_n. $\mathcal{O}_X(E^*)$ is a locally free G-sheaf of $\mathcal{O}_X$-modules, and rank $\mathcal{O}_X(E^*) = \dim E^*$. Note that $\dim E^*$ is uncountable whenever $\dim E = \infty$.

The following Proposition is a key tool for calculating the cohomology groups of $\mathcal{O}_X(E^*)$ and $\mathcal{O}_X(E)$. In order to formulate it we need a definition. Let

$$\cdots \xrightarrow{\nu_{n+1}} M_n \xrightarrow{\nu_n} M_{n-1} \xrightarrow{\nu_{n-1}} \cdots \xrightarrow{\nu_2} M_1 \longrightarrow 0 \tag{10.2}$$

be an inverse system of vector spaces. If $m > n$, $\nu_{n,m}$ denotes the map $\nu_{n+1} \circ \cdots \circ \nu_m$ from M_m to M_n. For each n let $\widehat{M_n} := \varprojlim_{m>n} M_n/\nu_{n,m}(M_m)$, the completion of M_n with respect to its filtration by the subspaces $\nu_{n,m}(M_m)$, $m > n$. Then the value of the first right derived functor of $\varprojlim$ on the system (10.2) is the vector space $\varprojlim^{(1)} M_n := \prod_n (\widehat{M_n}/M_n)$.

Proposition 10.3. *Let* $\cdots \xrightarrow{\zeta_{n+1}} \mathcal{F}_n \xrightarrow{\zeta_n} \mathcal{F}_{n-1} \xrightarrow{\zeta_{n-1}} \cdots \xrightarrow{\zeta_2} \mathcal{F}_1 \to 0$ *be an inverse system of sheaves of locally free* $\mathcal{O}_X$*-modules.*

(i) *For every* $q \geq 0$ *there is a canonical exact sequence*

$$0 \to \varprojlim^{(1)} H^{q-1}(X; \mathcal{F}_n) \to H^q(X; \varprojlim \mathcal{F}_n) \to \varprojlim H^q(X; \mathcal{F}_n) \to 0, \tag{10.4}$$

where (for $q = 0$*)* $\varprojlim^{(1)} H^{-1}(X; \mathcal{F}_n) := 0$.

(ii) *If for some* $q > 0$ *the inverse system of vector spaces*

$$\cdots \xrightarrow{\zeta^q_{n+1}} H^q(X; \mathcal{F}_n) \xrightarrow{\zeta^q_n} H^q(X; \mathcal{F}_{n-1}) \xrightarrow{\zeta^q_{n-1}} \cdots \longrightarrow 0$$

satisfies the Mittag-Leffler condition, i.e., if for every n the filtration on $H^q(X; \mathcal{F}_n)$ *by the subspaces* $\zeta^q_{n+1,m}(H^q(X; \mathcal{F}_m))$ *is eventually constant, then there is a canonical isomorphism*

$$H^q(X; \varprojlim \mathcal{F}_n) \cong \varprojlim H^q(X; \mathcal{F}_n). \tag{10.5}$$

(iii) *If the* $\mathcal{F}_n$*'s are* G_n*-sheaves and* ζ_n *are morphisms of* G_{n-1}*-sheaves, then* (10.4) *is an exact sequence of G-modules. In particular, if the Mittag-Leffler condition is satisfied,* (10.5) *is a G-module isomorphism.*

Proof. The statements (i) and (ii) are adaptations of [11, Theorem 4.5] to our situation, and we will not reproduce the proofs here. See also [10, O_{III}, 13.3.1], which is the original source for Theorem 4.5 in [11].

To prove (iii), note that the canonical surjection

$$H^q(X; \varprojlim \mathcal{F}_n) \to \varprojlim H^q(X; \mathcal{F}_n) \tag{10.6}$$

is induced (by universality of $\varprojlim H^q(X; \mathcal{F}_n)$) by the maps $H^q(X; \varprojlim \mathcal{F}_n) \longrightarrow H^q(X; \mathcal{F}_n)$, which in turn are induced by the canonical maps $\varprojlim \mathcal{F}_n \longrightarrow \mathcal{F}_n$. As the latter are morphisms of G_n-sheaves, the former is a compatible system of maps of G_n-modules, and this implies (iii). (In this way the space $\varprojlim^{(1)} H^{q-1}(X; \mathcal{F}_n)$ is equipped with the G-module structure defined by the surjection (10.6) by means of the exact sequence (10.4).) □

If E is a rational P-module then Proposition 10.3 applies to both sheaves $\mathcal{O}_X(E^*)$ and $\mathcal{O}_X(E)$. Consider $\mathcal{O}_X(E^*)$ first. We have $\mathcal{O}_X(E^*) = \varprojlim \mathcal{O}_{X_n}(E_n^*)$, and here the Mittag-Leffler condition is automatic because $\dim H^q(X; \mathcal{O}_{X_n}(E_n^*)) = \dim H^q(X_n; \mathcal{O}_{X_n}(E_n^*)) < \infty$ for all q and all n. To apply Proposition 10.3 to $\mathcal{O}_X(E)$ note that $\mathcal{O}_X(E) = \varprojlim \mathcal{O}_{X_n}(E)$, where $\mathcal{O}_{X_n}(E)$ is the sheaf of local sections of the G_n-homogeneous vector bundle $G_n \times_{P_n} E \to X_n$ associated to E. In this case the cohomology groups $H^q(X_n; \mathcal{O}_{X_n}(E))$ are in general infinite dimensional, and therefore the Mittag-Leffler condition is not automatic. In fact we will see in Section 16 below that the Mittag-Leffler condition fails for some quite straightforward irreducible P-modules E.

11. Statement of main results on cohomology of $\mathcal{O}_X(E^*)$. In this and the following sections, G is a fixed locally reductive ind-group, P is a fixed parabolic subgroup of G, $X = G/P$, and E is an irreducible rational P-module. We assume further that E is expressed as $\varinjlim E_n$ for a minimal direct system of P_n-module injections $E_n \hookrightarrow E_{n+1}$. This is possible by Proposition 8.3(i). Throughout Sections 11 through 15 we write ζ_n for the canonical restriction map $\mathcal{O}_{X_n}(E_n^*) \to \mathcal{O}_{X_{n-1}}(E_{n-1}^*)$.

These are our main results on the cohomology of the sheaf $\mathcal{O}_X(E^*)$.

THEOREM 11.1. (i) *For the group G and the P-module E specified above,*

(a) *$H^q(X; \mathcal{O}_X(E^*))$ is a pro-rational G-module, and more precisely, for every integer $q \geq 0$, there is a canonical G-module isomorphism $H^q(X; \mathcal{O}_X(E^*)) = \varprojlim H^q(X_n; \mathcal{O}_{X_n}(E_n^*))$;*

(b) *$H^0(X; \mathcal{O}_X(E^*)) \neq 0$ if and only if E is dominant. In this case there is a canonical G-module isomorphism $H^0(X; \mathcal{O}_X(E^*)) = V(E)^*$.*

(ii) *Under the additional assumption that E is locally irreducible, $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ for at most one integer $q_0 \geq 0$. If q_0 is such an integer then $H^{q_0}(X; \mathcal{O}_X(E^*)) = V^*$ for some locally irreducible rational G-module V.*

(iii) *Suppose that G is root-reductive and that E is irreducible but not necessarily locally irreducible. Then $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ for at most one q_0, and $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ if and only if $H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$ for large enough n. Furthermore, whenever $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$, there is a Borel subgroup B of P and an element w of $\mathcal{W}$ of length q_0 with respect to $\mathfrak{b}$ such that $H^{q_0}(X; \mathcal{O}_X(E^*)) \cong V({}^wE)^*$, where ${}^wE$ is an irreducible ${}^wP$-module unique up to isomorphism.*

12. Proof and discussion of (i) and (ii).

Proof of (i). Statement (a) is a direct corollary of Proposition 10.3(ii) and (iii). To prove (b) note first that, precisely as in the classical case, Frobenius Reciprocity holds for $H^0(X; \mathcal{O}_X(E^*))$: we have a canonical isomorphism

$$\mathrm{Hom}_G(W, H^0(X; \mathcal{O}_X(E^*))) = \mathrm{Hom}_P(W, E^*) \tag{12.1}$$

for any G-module W. Indeed, (12.1) follows immediately from (10.5) for $\mathcal{F} = \mathcal{O}_X(E^*)$ and $q = 0$, from the universality property of the inverse limit, and from Frobenius Reciprocity for $H^0(X_n; \mathcal{O}_{X_n}(E_n^*))$ for all n. Now since the right-hand side of (12.1) is nonzero for $W = V(E)^*$, $H^0(X; \mathcal{O}_X(E^*))$ is nonzero whenever E is dominant. Therefore, by (10.5), $H^0(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$ for large enough n, and by the classical Borel-Weil Theorem we have $H^0(X_n; \mathcal{O}_{X_n}(E_n^*))^* = V(E_n)$. Thus, for a dominant E, $\varinjlim H^0(X_n; \mathcal{O}_{X_n}(E_n^*))^* = \varinjlim V(E_n)$. But then, as in the first part of the proof of Proposition 7.4, there is a G-isomorphism $V(E) \cong \varinjlim V(E_n)$, and consequently $H^0(X; \mathcal{O}_X(E^*)) = V(E)^*$.

Conversely, let $H^0(X; \mathcal{O}_X(E^*)) = \varprojlim H^0(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$. Then $\varinjlim H^0(X_n; \mathcal{O}_{X_n}(E_n^*))^*$ is a well-defined G-module which admits a P-module injection $E \hookrightarrow$

$\varinjlim H^0(X_n; \mathcal{O}_{X_n}(E_n^*))^*$ (whose dual corresponds via (12.1) to the identity map $H^0(X; \mathcal{O}_X(E^*)) \to H^0(X; \mathcal{O}_X(E^*))$) and is moreover generated over G by the image of E. Again, as in the proof of Proposition 7.4 this implies that $\varinjlim H^0(X_n; \mathcal{O}_{X_n}(E_n^*))^*$ is canonically isomorphic to $V(E)$ as a G-module. In this way, whenever $H^0(X; \mathcal{O}_X(E^*)) \neq 0$, E is dominant and $H^0(X; \mathcal{O}_X(E^*)) = V(E)^*$. □

The above argument shows in particular that $V(E_n)$ is finite dimensional for each n whenever E is dominant. Therefore the proof Proposition 7.4 is now complete. Furthermore, Theorem 11.1(i)(b) and Proposition 8.5 have the following essential corollary.

COROLLARY 12.2. *Let G, P and E be as in Theorem 11.1(i). Suppose that E is a weight P-module. If $H^0(X; \mathcal{O}_X(E^*)) \neq 0$ then E is a finite (and thus locally irreducible) P-module, and $H^0(X; \mathcal{O}_X(E^*))$ is isomorphic to the dual of the rational locally irreducible finite G-module $V(E)$.*

Of course, the requirement that E be a weight module is crucial in Corollary 12.2. For if G, P, E and V are as in Example 8.6, then $H^0(X; \mathcal{O}_X(E^*)) = V^*$, and V is an irreducible but not locally irreducible G-module which is not a weight module.

Proof of (ii). Since $\mathcal{O}_{X_n}(E_n^*)$ is an irreducible G_n-homogeneous $\mathcal{O}_{X_n}$-module, by the classical Bott-Borel-Weil Theorem $H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$ for at most one integer q_0. Therefore $\varprojlim H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$ for at most one q_0, and this, combined with (i)(a), implies that $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ for at most one q_0. If now $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$, V is defined simply as the limit of the direct system of irreducible finite dimensional G_n-modules

$$0 \to H^{q_0}(X_0; \mathcal{O}_{X_0}(E_0^*))^* \to \cdots \to H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*))^* \to \cdots, \tag{12.3}$$

and is therefore a locally irreducible rational G-module. □

The problem left open by (ii) is that of an explicit description of the G-module V in terms of the P-module E. In the setting where G is root-reductive, (iii) is an essential step toward solving that problem. For general ind-groups the problem is open even in the case when $P = B$ is a Borel group and E is a one dimensional B-module E; see Example 14.5 below.

13. Proof and discussion of (iii).

Proof of (iii) for locally irreducible E. The fact that $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ for at most one q_0 was established in the proof of (ii). Assume, more generally, that $H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) \neq 0$ for some q_0 and for large enough n. (If no such q_0 exists, there is nothing to prove.) Then, for large enough n, all E_n are regular and $\ell(E_n) = q_0$. Furthermore, Proposition 9.7 yields a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{p}$ and an element $w \in \mathcal{W}$ of length q_0 with respect to $\mathfrak{b}$. Let λ_n is the $\mathfrak{b}_n$-highest weight of E_n and $\mu_n := w(\lambda_n + \rho_n) - \rho_n$. The Bott-Borel-Weil Theorem gives $H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) \cong V(\mathbb{C}_{\mu_n})^*$, where $\mathbb{C}_{\mu_n}$ denotes the one dimensional $\mathfrak{b}_n$-module of weight μ_n. As in the proof of (ii), we define V as the limit of the direct system dual to the system of projections $\zeta_{n+1}^{q_0}\colon\ H^{q_0}(X_{n+1}; \mathcal{O}_{X_{n+1}}(E_{n+1}^*)) \to H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*))$.

We claim that $V \neq 0$, i.e., that $\zeta_{n+1}^{q_0} \neq 0$ for large enough n. We will prove this by induction on q_0. If $q_0 = 0$, E is dominant by Proposition 7.4, and thus the statement follows from (i)(b). Let $q_0 > 0$ and let $w = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{q_0}}$ be a minimal decomposition of w as a product of $\mathfrak{b}_{n+1}$-simple reflections. Define $P_{n+1}^{\alpha_{q_0}}$ as the minimal proper parabolic subgroup of G_{n+1} such that $B_{n+1} \subset P_{n+1}^{\alpha_{q_0}}$ and $\pm\alpha_{q_0} \in \Delta_{\mathfrak{p}_{n+1}^{\alpha_{q_0}}}$. Then $P_n^{\alpha_{q_0}} := P_{n+1}^{\alpha_{q_0}} \cap G_n$ is a parabolic subgroup of G_n, and there is a commutative diagram

$$\begin{array}{ccc} G_n/B_n & \hookrightarrow & G_{n+1}/B_{n+1} \\ p_n^{\alpha_{q_0}} \downarrow & & \downarrow p_{n+1}^{\alpha_{q_0}} \\ G_n/P_n^{\alpha_{q_0}} & \hookrightarrow & G_{n+1}/P_{n+1}^{\alpha_{q_0}}. \end{array}$$

Notice first that $\mathcal{O}_{X_{n+1}}(E_{n+1}^*) = (\pi_{n+1})_* \mathcal{O}_{G_{n+1}/B_{n+1}}(\mathbb{C}_{-\lambda_{n+1}})$ and $\mathcal{O}_{X_n}(E_n^*) = (\pi_n)_* \mathcal{O}_{G_n/B_n}(\mathbb{C}_{-\lambda_n})$, where $\pi_{n+1}\colon\ G_{n+1}/B_{n+1} \to X_{n+1}$ and $\pi_n\colon\ G_n/B_n \to X_n$ are the canonical projections. Demazure's proof [6] of the Bott-Borel-Weil Theorem implies

$$\begin{aligned} H^{q_0}(X_{n+1}; \mathcal{O}_{X_{n+1}}(E_{n+1}^*)) &= H^{q_0}(G_{n+1}/B_{n+1}; \pi_{n+1}^* \mathcal{O}_{X_{n+1}}(E_{n+1}^*)) \\ &= H^{q_0-1}(G_{n+1}/P_{n+1}^{\alpha_{q_0}}; R^1(p_{n+1}^{\alpha_{q_0}})_*(\pi_{n+1}^* \mathcal{O}_{X_{n+1}}(E_{n+1}^*))) \end{aligned}$$

and

$$\begin{aligned} H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*)) &= H^{q_0}(G_n/B_n; \pi_n^* \mathcal{O}_{X_n}(E_n^*)) \\ &= H^{q_0-1}(G_n/P_n^{\alpha_{q_0}}; R^1(p_n^{\alpha_{q_0}})_*(\pi_n^* \mathcal{O}_{X_n}(E_n^*))). \end{aligned}$$

Thus, for the induction step it suffices to check that $\zeta_{n+1}\colon\ \mathcal{O}_{X_{n+1}}(E_{n+1}^*) \to \mathcal{O}_{X_n}(E_n^*)$ induces a nonzero G_n-morphism

$$\tilde{\zeta}_{n+1}^1\colon\ R^1(p_{n+1}^{\alpha_{q_0}})_*(\pi_{n+1}^* \mathcal{O}_{X_{n+1}}(E_{n+1}^*)) \to R^1(p_n^{\alpha_{q_0}})_*(\pi_n^* \mathcal{O}_{X_n}(E_n^*)).$$

Indeed, the observation that $\zeta_{n+1}^{q_0} = (\tilde{\zeta}_{n+1}^1)^{q_0-1}$ will then allow us to conclude that $\zeta_{n+1}^{q_0} \neq 0$ as $(\tilde{\zeta}_{n+1}^1)^{q_0-1}$ is nonzero by the induction assumption. But the fact that

$\tilde{\zeta}^1_{n+1} \neq 0$ follows immediately from the long exact sequence

$$\begin{aligned} 0 \;\to\; & (p^{\alpha_{q_0}}_{n+1})_* \ker \zeta_{n+1} \to (p^{\alpha_{q_0}}_{n+1})_* \mathcal{O}_{X_{n+1}}(E^*_{n+1}) \to (p^{\alpha_{q_0}}_n)_* \mathcal{O}_{X_n}(E^*_n) \\ \to\; & R^1(p^{\alpha_{q_0}}_{n+1})_* \ker \zeta_{n+1} \to R^1(p^{\alpha_{q_0}}_{n+1})_* \mathcal{O}_{X_{n+1}}(E^*_{n+1}) \to R^1(p^{\alpha_{q_0}}_n)_* \mathcal{O}_{X_n}(E^*_n) \to 0. \end{aligned}$$

Therefore finally $\zeta^{q_0}_{n+1} \neq 0$, and thus also $V \neq 0$.

It remains to show that V admits an irreducible ${}^w\mathfrak{p}$-submodule ${}^wE$. Then $V \cong V({}^wE)$ by the remark after Proposition 7.3, and the uniqueness of ${}^wE$ follows from the fact that ${}^wE$ is the subspace of $\mathfrak{u}_{{}^w\mathfrak{p}}$-invariants in $V({}^wE)$. To define ${}^wE$ set ${}^wE := \varinjlim U({}^w\mathfrak{p}_n) \cdot V(\mathbb{C}_{\mu_n})^{\mu_n}$. As $V(\mathbb{C}_{\mu_n})^{\mu_n}$ is the $\mathfrak{b}_n$-highest weight space of $V(\mathbb{C}_{\mu_n})$, $U({}^w\mathfrak{p}_n) \cdot V(\mathbb{C}_{\mu_n})^{\mu_n}$ is an irreducible ${}^w\mathfrak{p}_n$-submodule of $V(\mathbb{C}_{\mu_n})$. To show that ${}^wE$ is well defined, we need to check only that $(\zeta^{q_0}_{n+1})^*(U({}^w\mathfrak{p}_n) \cdot V(\mathbb{C}_{\mu_n})^{\mu_n}) \subset U({}^w\mathfrak{p}_{n+1}) \cdot V(\mathbb{C}_{\mu_{n+1}})^{\mu_{n+1}}$, i.e., that $(\zeta^{q_0}_{n+1})^*(V(\mathbb{C}_{\mu_n})^{\mu_n}) \subset U({}^w\mathfrak{p}_{n+1}) \cdot V(\mathbb{C}_{\mu_{n+1}})^{\mu_{n+1}}$. Fix a large enough n and let $\kappa_1, \ldots, \kappa_s$ be all weights of $V(\mathbb{C}_{\mu_{n+1}})^{\mu_{n+1}}$ onto which $(\zeta^{q_0}_{n+1})^*(V(\mathbb{C}_{\mu_n})^{\mu_n})$ projects nontrivially. (As V is not necessarily a weight module, s may be greater than one.) Then $\kappa_{i|\mathfrak{h}_n} = w(\lambda_n + \rho_n) - \rho_n$ for all i. But since $\mu_{n+1} = w(\lambda_{n+1} + \rho_{n+1}) - \rho_{n+1}$, an immediate verification shows that all κ_i are of the form

$$\mu_{n+1} + (\mathbb{Z}_+\text{-linear combination of } \mathfrak{b}_{n+1}\text{-negative roots of } {}^w\mathfrak{p}_{n+1}).$$

Therefore, indeed $(\zeta^{q_0}_{n+1})^*(V(\mathbb{C}_{\mu_n})^{\mu_n}) \subset U({}^w\mathfrak{p}_{n+1}) \cdot V(\mathbb{C}_{\mu_{n+1}})^{\mu_{n+1}}$, and the proof is complete. □

Note that (iii) has the following combinatorial consequence. The ${}^w\mathfrak{p}_n$-modules ${}^wE_n$ satisfy ${}^wE_n \prec {}^wE_m$ for large enough $m > n$. This fact can be checked directly in the spirit of Proposition 9.7, but the construction of the ${}^wP$-module ${}^wE$ gives it as a trivial corollary. Furthermore, the modules ${}^wE_n$ are defined simply by the sequence of modules $\{E_n\}$, however in general the injections ${}^wE_n \to {}^wE_{n+1}$ (and hence the module ${}^wE$) depend on the $\mathfrak{p}$-module E as a whole. In the special case when every infinite dimensional $\mathfrak{l}^s$ in the decomposition (4.4) of every simple $\mathfrak{g}^t$, cf. Proposition 4.1, is isomorphic to $\mathfrak{gl}(\infty)$, both E and ${}^wE$ are determined by the sequence $\{E_n\}$ only, and are automatically weight modules. The reason is that the branching of an irreducible finite dimensional $\mathfrak{gl}(n+1)$-module into irreducible $\mathfrak{gl}(n)$-modules is multiplicity-free.

The next example shows that ${}^wE$ may not admit an irreducible submodule of any parabolic subgroup which is properly contained in ${}^wP$, in particular, of P. Furthermore, if ${}^wP = G$ then ${}^wE = V$. As a consequence, even when E is a finite weight P-module, V need not be a finite G-module as Proposition 8.5 no longer applies. See Example 13.1(2) for a nonfinite weight G-module V such that $V^* \cong H^2(X; \mathcal{O}_X(E^*))$. Note also that in general E and ${}^wE$ are not determined by the weight sequence $\{\lambda_n\}$.

Example 13.1.

(1) Let $G = GL(\infty)$, let P be as in Example 4.3, and let $B \subset P$ be the Borel subgroup of all upper triangular matrices in G. So $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j\}$ is the set of roots of $\mathfrak{b}$. Define the P-module E as the direct limit of irreducible P_n-modules with B_n-highest weights $\lambda_n := -\varepsilon_1 + n\varepsilon_2$. The branching rule ensures that E is well defined as the respective inclusions are unique (up to a multiplicative constant). Then $H^1(X; \mathcal{O}_X(E^*)) \cong V^*$, where V is the direct limit of the irreducible G_n-modules with B_n-highest weights $\mu_n = (n-1)\varepsilon_1$. The assertion of Theorem 11.1(iii) is satisfied with $w = \sigma_\alpha$ for $\alpha = \varepsilon_1 - \varepsilon_2$. Here ${}^wP = G$; see Example 4.3 above. It is not difficult to check that V is a finite weight G-module that has no P'-submodule that is irreducible for any proper parabolic subgroup $P' \subset G$.

(2) Now let $\mathfrak{p} \subset \mathfrak{gl}(\infty)$ be the parabolic subalgebra with roots

$$\{\varepsilon_i - \varepsilon_j \mid 3 \leq i, 3 \leq j, i \neq j\} \cup \{\varepsilon_1 - \varepsilon_i, \varepsilon_i - \varepsilon_2 \mid 3 \leq i\} \cup \{\varepsilon_1 - \varepsilon_2\},$$

let $\mathfrak{b} \subset \mathfrak{p}$ be the Borel subalgebra with roots

$$\begin{aligned}\{\varepsilon_i - \varepsilon_j \mid 5 \leq i, 5 \leq j, i < j\} \;&\cup\; \{\varepsilon_1 - \varepsilon_i, \varepsilon_3 - \varepsilon_i, \varepsilon_i - \varepsilon_2, \varepsilon_i - \varepsilon_4 \mid 5 \leq i\} \\ &\cup\; \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_4, \varepsilon_3 - \varepsilon_2, \\ &\qquad \varepsilon_3 - \varepsilon_4, \varepsilon_4 - \varepsilon_2\},\end{aligned}$$

and let P and B be the corresponding parabolic and Borel subgroups of $G = GL(\infty)$. Define the P-module E as the direct limit of the irreducible P_n-modules with B_n-highest weights $\lambda_n := -\varepsilon_1 + \varepsilon_2 + n\varepsilon_3 - n\varepsilon_4$. Then $H^2(X; \mathcal{O}_X(E^*)) \cong V^*$, where V is the direct limit of the irreducible G_n-modules with B_n-highest weights $\mu_n = (n-1)\varepsilon_1 + (1-n)\varepsilon_2$. Here the assertion of Theorem 11.1(iii) is satisfied with $w = \sigma_\alpha \circ \sigma_\beta$ for $\alpha = \varepsilon_1 - \varepsilon_3$ and $\beta = \varepsilon_2 - \varepsilon_4$, and again ${}^wP = G$. A direct verification shows that neither is E a finite P-module nor is V a finite G-module. Therefore V^* cannot appear as the zeroth cohomology of $\mathcal{O}_X(E'^*)$ for any rational P-module E'.

Proof of (iii) for general E. Note first that, as $\mathfrak{u}_\mathfrak{p} \cdot E = 0$ (Proposition 7.1), E is an irreducible $\mathfrak{p}^{\mathrm{red}}$-module and is in particular completely reducible as $\mathfrak{p}_n^{\mathrm{red}}$-module for all n. Let $E_n = \bigoplus_{i=1}^{i_n} E_n^i$ be the decomposition of E_n into isotypic $\mathfrak{p}_n^{\mathrm{red}}$-components. Fix furthermore a decomposition of each E_n^i as $\bigoplus_k E_n^{i,k}$, $E_n^{i,k}$ being simple. For any sequence of simple $\mathfrak{p}_n^{\mathrm{red}}$-components $E_n^{i_n,k_n}$, let $f_{m,n}\colon\ E_n^{i_n,k_n} \to E_m^{i_m,k_m}$ denote the composition of the injection $E_n^{i_n,k_n} \hookrightarrow E_m$ followed by the projection $E_m \to E_m^{i_m,k_m}$ determined by the fixed decomposition of $E_m^{i_m}$.

For any q, $H^q(X_n; \mathcal{O}_{X_n}(E_n^*)) = \bigoplus_i H^q(X_n; \mathcal{O}_{X_n}((E_n^i)^*)) = \bigoplus_{i,k} H^q(X_n;\ \mathcal{O}_{X_n}((E_n^{i,k})^*))$. If for no q $H^q(X_n; \mathcal{O}_{X_n}((E_n)^*)) \neq 0$ for all large enough n, $H^q(X; \mathcal{O}_X((E)^*)) = 0$ for all q, and there is nothing to prove. Assume therefore that there is

q_0 with $H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n)^*)) \neq 0$ for all large enough n. By the Bott-Borel-Weil Theorem this is the same as assuming that for large enough n some isotypic component $E_n^{i_0}$ is regular and $\ell(E_n^{i_0}) = q_0$.

We claim that then, for large enough n, every isotypic component E_n^i is regular and $\ell(E_n^i) = q_0$. Assume not. We will construct a locally irreducible locally finite $\mathfrak{p}$-module E' whose existence contradicts Proposition 9.7(ii). Let $E_{n_1}^{i_1,k_1}$ be regular of length q_0. Find $E_{n_2}^{i_2,k_2}$ with $n_2 > n_1$ which is either not regular or not of length q_0 and such that the corresponding f_{n_2,n_1} is nonzero. $E_{n_2}^{i_2,k_2}$ exists because of our assumption and the minimality of the system of injections $E_n \hookrightarrow E_{n+1}$. Furthermore, let $E_{n_3}^{i_3,k_3}$ be regular of length q_0 with $n_3 > n_2$ and $f_{n_3,n_2} \neq 0$. $E_{n_3}^{i_3,k_3}$ exists also due to the minimality of the system of injections $E_n \hookrightarrow E_{n+1}$. Continuing in this way, we construct $E' := \varinjlim_j E_{n_j}^{i_j,k_j} \neq 0$, where $E_{n_j}^{i_j,k_j}$ is regular of length q_0 for odd j and is not such for even j. The existence of E' is contradicts Proposition 9.7(ii), therefore, for large enough n, each isotypic component E_n^i is regular and $\ell(E_n^i) = q_0$. In particular, $H^q(X; \mathcal{O}_X(E^*)) = 0$ for $q \neq q_0$, i.e., $H^{q_0}(X; \mathcal{O}_X(E^*))$ is the only possibly nonvanishing cohomology group of $\mathcal{O}(E^*)$.

In a similar way we construct a Borel subgroup $B \subset P$ and an element $w \in \mathcal{W}$ of length q_0 with respect to $\mathfrak{b}$. Indeed, for a sequence of simple components $E_n^{i_n,k_n}$ such that $\varinjlim E_n^{i_n,k_n} \neq 0$, Proposition 9.7(ii) yields a Borel subgroup B of P and an element $w \in \mathcal{W}$ of length q_0 with respect to $\mathfrak{b}$. We claim that any sequence of simple components $E_m^{i_m,k_m}$ with $\varinjlim E_m^{i_m,k_m} \neq 0$ yields the same Borel subgroup and the same element $w \in \mathcal{W}$, as otherwise, a construction similar to the above would give a module whose existence is contradictory.

We are ready to complete the argument. As in the proof of (ii), define the rational G-module V as the limit of the direct system (12.3). Note that, as all isotypic components E_n^i are regular for large enough n, and, as the length of w with respect to $\mathfrak{b}$ is q_0, the decomposition $H^{q_0}(X_n; \mathcal{O}_{X_n}(E_n^*))^* = \bigoplus_i H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n^i)^*))^*$ is a decomposition into $\mathfrak{g}_n$-isotypic components. Furthermore, $H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n^{i,k})^*))^*$ are the simple $\mathfrak{g}_n$-components of $H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n^i)^*))^*$. The crucial observation is that our argument for (iii) in the locally irreducible case implies that the multiplicity diagram D_V of V, defined by the decompositions $\bigoplus_i H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n^i)^*))^* = \bigoplus_{i,k} H^{q_0}(X_n; \mathcal{O}_{X_n}((E_n^{i,k})^*))^*$, is identical to the multiplicity diagram of E considered as a $\mathfrak{p}^{\mathrm{red}}$-module. Therefore $V \neq 0$. Moreover, by Proposition 8.3(iii), D_V admits no nonzero stably proper subdiagram. Consequently, V is irreducible by Proposition 8.3(ii). It remains to show that $V = V({}^wE)$ for a unique irreducible ${}^wP$-submodule ${}^wE$ of V. If $w \in \mathcal{W}_{n_0}$ for some n_0, the irreducible ${}^WP_n$-module ${}^wE_n^i$ is well defined for any E_n^i with $n > n_0$. Furthermore, as we showed when proving (iii) in the locally irreducible case, an injection $E_n^i \to E_{n+1}^j$ induces an injection ${}^wE_n^i \to {}^wE_{n+1}^j$. Hence, for $n \geq n_0$, there is a well-defined G_n-module injection of ${}^wE_n := \bigoplus_i {}^w(E_n^i)$ into ${}^wE_{n+1} := \bigoplus_i {}^w(E_{n+1}^i)$, i.e. ${}^wE := \varinjlim {}^wE_n$ is a well-defined ${}^wP$-submodule of V. The irreducibility of V yields the irreducibility of ${}^wE$, the uniqueness of ${}^wE$, and the existence of a canonical isomorphism $V({}^wE) = V$. □

14. The situation when E has a highest weight. The case when the P-module E has a highest weight with respect to some Borel subgroup $B \subset P$ is a close analog of the classical Bott-Borel-Weil Theorem. We show below that, when E has a highest weight, statement (iii) of Theorem 11.1 can be reformulated in very familiar terms without reference to the parabolic subgroup ${}^wP$. Of course this applies to the case when E is finite dimensional, in particular to the case when $P = B$ and $\dim E = 1$.

PROPOSITION 14.1. *Suppose that G is root-reductive and E has highest weight λ with respect to a Borel subgroup $B \subset P$. Let Δ^+ denote the roots of $\mathfrak{b}$. Then $H^{q_0}(X; \mathcal{O}_X(E^*)) \neq 0$ if and only if there exists $w \in \mathcal{W}$ of length q_0 with respect to $\mathfrak{b}$, such that $\mu := w(\lambda) - (\sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+} \alpha)$ is a B-dominant integral weight. In this case*

$$H^{q_0}(X; \mathcal{O}_X(E^*)) = V(\mathbb{C}_\mu)^*, \quad H^q(X; \mathcal{O}_X(E^*)) = 0 \quad \textit{for } q \neq q_0, \tag{14.2}$$

where $\mathbb{C}_\mu$ is the one dimensional B-module of weight μ.

Proof. Under the hypotheses of the proposition, there is a canonical G-isomorphism

$$H^{\cdot}(X; \mathcal{O}_X(E^*)) \cong H^{\cdot}(G/B; \mathcal{O}_{G/B}(\mathbb{C}_{-\lambda})). \tag{14.3}$$

Indeed, the classical Bott-Borel-Weil Theorem yields a compatible family of G_n-isomorphisms

$$H^{\cdot}(X_n; \mathcal{O}_{X_n}(E_n^*)) \cong H^{\cdot}(G_n/B_n; \mathcal{O}_{G_n/B_n}(\mathbb{C}_{-\lambda_n})), \tag{14.4}$$

where $\lambda_n := \lambda|_{\mathfrak{h}_n}$, and (14.3) follows from Theorem 11.1(i)(a) via (14.4).

Suppose that there exists $w \in \mathcal{W}$ of length q_0 as above. Then $\mu_n := \mu|_{\mathfrak{h}_n}$ is a B_n-dominant weight for each n. The Bott-Borel-Weil Theorem, together with Proposition 7.4 and Theorem 11.1, gives

$$\begin{aligned} H^{q_0}(G/B; \mathcal{O}_{G/B}(\mathbb{C}_{-\lambda})) &= \varprojlim H^{q_0}(G_n/B_n; \mathcal{O}_{G_n/B_n}(\mathbb{C}_{-\lambda_n})) \\ &= \varprojlim V(\mathbb{C}_{\mu_n})^* = V(\mathbb{C}_\mu)^*, \end{aligned}$$

and

$$H^q(G/B; \mathcal{O}_{G/B}(\mathbb{C}_{-\lambda})) = 0 \text{ for } q \neq q_0.$$

This combines with (14.3) to give (14.2).

Conversely, suppose $H^{q_0}(G/B; \mathcal{O}_{G/B}(\mathbb{C}_{-\lambda})) = \varprojlim H^{q_0}(G_n/B_n; \mathcal{O}_{G_n/B_n}(\mathbb{C}_{-\lambda_n})) \neq 0$. Then, by the Bott-Borel-Weil Theorem, the $\mathfrak{b}_n$-module $\mathbb{C}_{\lambda_n}$ has length q_0 for n sufficiently large. Apply Proposition 9.7 to $E = \mathbb{C}_\lambda$ with $\mathfrak{p} = \mathfrak{b}$ for the existence of $w \in \mathcal{W}$ as required. Now (14.2) holds by the same argument as above. $\square$

The following examples illustrate the problem of describing the G-module structure on the only nonvanishing cohomology $H^{q_0}(X; \mathcal{O}_X(E^*))$ for more general ind-groups G in the special case when $P = B$ and E is one dimensional.

Example 14.5.

(1) This example is similar to an example in [20]. Let $G = GL(2^\infty)$ and let $P = B$ be the Borel subgroup of upper triangular matrices. Given an integer $q_0 \geq 0$, it is an interesting combinatorial problem to describe all systems of integral weights $\{\lambda_n\}$ (λ_n being a weight of $\mathfrak{g}_n = \mathfrak{gl}(2^n)$) such that λ_n projects onto λ_{n-1}, and $H^{q_0}(X_n; \mathcal{O}_{X_n}(\mathbb{C}_{-\lambda_n})) \neq 0$ for large enough n. Another natural question is whether, given such a weight system, it always yields a nonzero cohomology $H^{q_0}(X; \mathcal{O}_X(E^*))$ for $E := \varinjlim \mathbb{C}_{\lambda_n}$, and whether $H^{q_0}(X; \mathcal{O}_X(E^*))$ is always the dual of a $\mathfrak{b}$-highest weight module. Here is a straightforward way to produce weight systems $\{\lambda_n\}$ as above for any $q_0 \geq 0$. Fix $n_0 \geq 1$ and an integral weight $\lambda_{n_0} =: \sum_{1 \leq i \leq 2^{n_0}} \lambda^i \varepsilon_{i,n_0}$ such that $H^{q_0}(X_{n_0}; \mathcal{O}_{X_{n_0}}(\mathbb{C}_{-\lambda_{n_0}})) \neq 0$. Let c be a nonpositive integer, less or equal to $\min\{\lambda^1, \ldots, \lambda^{2^{n_0}}\}$. For $n > n_0$, define the weight λ_n recursively by

$$\lambda_n := \lambda_{n-1} - c \sum_{1 \leq i \leq 2^{n-1}} \varepsilon_{i,n} + c \sum_{2^{n-1}+1 \leq j \leq 2^n} \varepsilon_{j,n}.$$

A trivial verification shows that $\{\lambda_n\}$ is a system of integral weights as desired. Set $E := \varinjlim \mathbb{C}_{\lambda_n}$. An easy computation based on the Bott-Borel-Weil Theorem implies that $H^q(X; \mathcal{O}_X(E^*)) = 0$ for $q \neq q_0$, and

$$H^{q_0}(X; \mathcal{O}_X(E^*)) = V(\tilde{E})^*,$$

where the B-module $\tilde{E}$ equals $\varinjlim_{n \geq n_0} \mathbb{C}_{w(\lambda_n + \rho_n) - \rho_n}$ (and $w \in S_{2^{n_0}}$ permutes only the first 2^{n_0} coordinates of $\lambda_n + \rho_n$).

(2) Here is another interesting situation in which there are no nonzero higher cohomology groups. Let $G = PGL^{\mathrm{Ad}}(k^{2^\infty})$, let $P = B$ be any Borel subgroup as described in Section 5, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We claim that in this situation all cohomology groups of $\mathcal{L}$ equal zero unless $\mathcal{L} \cong \mathcal{O}_X$.

The inverse images of $\mathcal{L}$ on X_n determine a sequence of integral $\mathfrak{gl}(k^{2^n})$-weights λ_n, and $\mathcal{L} \cong \mathcal{O}_X$ if and only if $\lambda_n = 0$ for all n. Assuming that the λ_n are not all zero, and assuming that $H^{q_0}(X; \mathcal{L}) \neq 0$ for some $q_0 \geq 0$, we show (following an idea of [1]) that, for n sufficiently large, $\lambda_n(\alpha_n) > \lambda_{n+1}(\alpha_{n+1})$, where α_n denotes the highest root of B_n. This computation is based on the explicit form of the immersions θ_n: $X_n \hookrightarrow X_{n+1}$; see Example 10.1(2) above. The details are left to the reader. In this way we obtain an infinite strictly decreasing sequence of positive integers, i.e., a contradiction. Therefore $\lambda_n = 0$ for all n, so $\mathcal{L} \cong \mathcal{O}_X$, $H^0(X; \mathcal{L}) = \mathbb{C}$, and $H^q(X; \mathcal{L}) = 0$ for $q > 0$.

There are uncountably many nonisomorphic G-homogeneous invertible sheaves of the form $\mathcal{O}_X(E^*)$. For any such sheaf, the above result implies $H^q(X;\mathcal{O}_X(E^*)) = 0$ for all $q > 0$, and that $H^0(X;\mathcal{O}_X(E^*)) \neq 0$ only when E is the trivial B-module. This latter fact is closely related to the discovery of Yu. Bahturin and G. Benkart, [1], that $\mathfrak{g} = \mathfrak{pgl}^{\mathrm{Ad}}(k^{2^\infty})$ does not admit locally finite highest weight modules except the trivial one. Indeed, if $H^0(X;\mathcal{O}_X(E^*))$ were nonzero for some nontrivial one dimensional B-module E, we would have had $H^0(X;\mathcal{O}_X(E^*)) = V^*$ for some highest weight module V, $V \ncong \mathbb{C}$. Therefore the result of Bahturin and Benkart implies the part of our result concerning $H^0(X;\mathcal{O}_X(E^*))$.

15. Projectivity of G/P. We are now ready to address the question of whether the ind-variety $X = G/P$ is projective. In general, X is not projective as one can see from Example 14.5(2). Moreover, for $G = PGL^{\mathrm{Ad}}(k^{2^\infty})$ and for any proper parabolic subgroup P that contains a Borel subgroup as in Section 5, X is never projective. For the existence of a very ample invertible sheaf $\mathcal{L}_X$ on X would imply the existence of an invertible sheaf $\mathcal{L} \ncong \mathcal{O}_{G/B}$ on G/B with $H^0(G/B;\mathcal{L}) \neq 0$.

On the other hand, the ind-variety $X = G/P$ from Example 10.1(2) (here $G = GL(2^\infty)$) is projective. A straightforward verification shows that, in this case, for any fixed n, every very ample invertible $\mathcal{O}_{X_n}$-module admits an extension to an invertible $\mathcal{O}_X$-module whose restrictions to X_m are very ample for all $m > n$.

For a root-reductive ind-group G we will give now an explicit projectivity criterion for $X = G/P$. Assuming that G is root-reductive, we have $X \cong \overset{\mathrm{f}}{\times}_{t\in T} G^t/P^t$, and the reader can check that X is projective if and only if G^t/P^t is projective for each t. Therefore it suffices to give a projectivity criterion for X under the assumption that G is classical simple.

PROPOSITION 15.1. *Let G be classical simple. Then $X = G/P$ is projective if and only if as an ordered set $S_{\mathfrak{p}}$ (see Section 4) is isomorphic to a subset of $\mathbb{Z}$ for $G = A(\infty)$, and to a subset of $\mathbb{Z}_-$ otherwise.*

Proof. Fix a Borel subgroup $B \subset P$. If $S_{\mathfrak{p}}$ is isomorphic to a subset of $\mathbb{Z}$ for $G = A(\infty)$, or to a subset of $\mathbb{Z}_-$ for $G = B(\infty)$, $C(\infty)$, $D(\infty)$, there exists a strictly decreasing function $\varphi\colon S_{\mathfrak{p}} \to \mathbb{Z}$ such that, if $G \neq A(\infty)$, $\varphi(s) \geq 0$ and $\varphi(s_0) = 0$ whenever $s_0 \in S_{\mathfrak{p}}$. Using φ, we will now determine a B-dominant integral weight λ as follows. First note that, just by the construction of $S_{\mathfrak{p}_n}$, each simple root of B_n has a well-defined image in $S_{\mathfrak{p}_n}$. Given $\alpha \in \Delta^+$, let $\alpha = c_n^1\alpha_n^1 + \cdots + c_n^{k_n}\alpha_n^{k_n}$, $c_n^j \neq 0$, be its decomposition into a linear combination of simple roots of B_n. (Here the images $s_n^1, \ldots, s_n^{k_n} \in S_{\mathfrak{p}_n}$ of $\alpha_n^1, \ldots, \alpha_n^{k_n}$ are in decreasing order.) To define λ, we set

$$\lambda(\alpha) := \begin{cases} \varphi(s_n^1) - \varphi(s_n^{k_n}) & \text{if} \quad c_n^j = 1 \quad \text{for} \quad i = j, \dots, k_n, \\ \varphi(s_n^1) + \varphi(s_n^{p_n}) & \text{if} \quad c_n^j = 1 \quad \text{for} \quad j = 1, \dots, k_p - 1, \quad c_n^{k_p} = 2, \\ 2\varphi(s_n^1) & \text{if} \quad c_1 = 2. \end{cases}$$

A direct verification shows that $\lambda(\alpha)$ does not depend on n for large enough n. Therefore λ is a well-defined integral weight. Furthermore, the definition of λ implies that $\mathbb{C}_\lambda$ is a well-defined dominant one dimensional P-module. Using the fact that $S_\mathfrak{p}$ is isomorphic to a subset of $\mathbb{Z}$, for $G = A(\infty)$, and to a subset of $\mathbb{Z}_-$ otherwise, the reader will check that the restriction of the sheaf $\mathcal{O}_X(\mathbb{C}_{-\lambda})$ to X_n is very ample for every n. Thus the sheaf $\mathcal{O}_X(\mathbb{C}_\lambda)$ provides a closed immersion of X into $\mathbb{P}(V(\mathbb{C}_\lambda))$.

Conversely, let X be projective. Fix a closed immersion i: $X \hookrightarrow \mathbb{P}(\mathbb{C}^\infty)$. The restriction $\mathcal{O}_{\mathbb{P}(\mathbb{C}^\infty)}(1)_{|X_n}$ is a very ample invertible $\mathcal{O}_{X_n}$-module. By choosing a G_n-linearization of $\mathcal{O}_X(1)_{|X_n}$ for each n (and by changing the signs of the corresponding weights) we obtain a system of integral weights $\{\lambda_n\}$ of $\mathfrak{g}_n$. This system is not canonical, but the integers $\lambda_n(\alpha)$ for $\alpha \in \Delta_n$ are canonical, i.e., depend only on the choice of the closed immersion i. The crucial observation now is that, if $S_\mathfrak{p}$ is not isomorphic to a subset of $\mathbb{Z}$ for $G = A(\infty)$, or to a subset of $\mathbb{Z}_-$ for $G = B(\infty)$, $C(\infty)$, $D(\infty)$, there always exists $\alpha \in \Delta\backslash\Delta_P$ such that for every r there are $\alpha_1, \alpha_2, \dots, \alpha_r \in \Delta\backslash\Delta_P$ with $\alpha = \alpha_1 + \dots + \alpha_r$. Fix n_0 for which $\alpha \in \Delta_{n_0}$. Set $r := \lambda_{n_0}(\alpha) + 1$ and let $n \geq n_0$ be such that $\alpha_1, \dots, \alpha_r \in \Delta_n$. Then $\lambda_{n_0}(\alpha) = \lambda_n(\alpha) = \sum_{i=1}^r \lambda_n(\alpha_i) \geq r = \lambda_{n_0}(\alpha) + 1$. This contradiction completes the proof. $\square$

16. The sheaf $\mathcal{O}_X(E)$ for infinite dimensional E. If E is finite dimensional, then so is E^*, and $\mathcal{O}_X(E) = \mathcal{O}_X((E^*)^*)$. Therefore in this case the consideration of $\mathcal{O}_X(E)$ does not lead to anything new. The situation is very different for an infinite dimensional E. Here the Mittag-Leffler condition can fail, and $\mathcal{O}_X(E)$ has in general arbitrarily many nonvanishing cohomology groups. It is an interesting problem to study the structure of these G-modules.

In the following example $\varprojlim^{(1)} H^0(X_n; \mathcal{O}_{X_n}(E_n)) \neq 0$, and $\varprojlim H^i(X_n; \mathcal{O}_{X_n}(E_n)) \neq 0$ for all i in a prescribed finite interval. The interested reader can construct a similar example in which $\varprojlim^{(1)} H^i(X_n; \mathcal{O}_{X_n}(E_n)) \neq 0$ for any i, and therefore $H^j(X; \mathcal{O}_X(E)) \neq 0$ for all $j > 0$.

Example 16.1.

(1) Let $B \subset P \subset G$ be as in Example 13.1(1). We define the irreducible rational P-module E as follows. Let $1 < k \in \mathbb{Z}_+$ and let E' be the irreducible rational P-module with B-highest weight $2k\varepsilon_2 + (2k-1)\varepsilon_3 + \dots + (k+1)\varepsilon_{k+1}$. Then

$E' := \varinjlim_{n>k} E'_n$ where E'_n is the irreducible P_n-module with B_n-highest weight $2k\varepsilon_2 + (2k-1)\varepsilon_3 + \cdots + (k+1)\varepsilon_{k+1}$. For $n > k$, $(E'_n)^*$ is a dominant P_n-module. Furthermore, there is a canonical P_n-module surjection $E'_{n+1} \to E'_n$, and thus a P_n-module injection $(E'_n)^* \hookrightarrow (E'_{n+1})^*$. Set $E_n := (E'_n)^*$ and $E := \varinjlim_{n>k} E_n$. Consider $\mathcal{O}_X(E) := \varprojlim \mathcal{O}_{X_n}(E)$, $\zeta_n\colon\ \mathcal{O}_{X_n}(E) \to \mathcal{O}_{X_{n-1}}(E)$ being the restriction maps. We claim that $\varprojlim^{(1)} H^0(X_n; \mathcal{O}_{X_n}(E)) \neq 0$, and furthermore that $H^q(X; \mathcal{O}_X(E)) \neq 0$ for $1 \leq q \leq k$. In particular, for $\mathcal{F}_n = \mathcal{O}_{X_n}(E)$, the exact sequence (10.4) has three nonzero terms when $q = 1$.

Consider $\varprojlim^{(1)}\ H^0(X_n; \mathcal{O}_{X_n}(E))$. When $n > k$, $H^0(X_n; \mathcal{O}_{X_n}(E))$ is the space E^{P_n} of P_n-invariants on E. Here $\dim E^{P_n} = \infty$, for as a P_n-module E is isomorphic to a direct sum of finitely many nontrivial irreducible P_n-modules and infinitely many copies of the trivial one dimensional P_n-module. Furthermore, the natural injection $E^{P_{n+1}} \subset E^{P_n}$ is immediately seen to be a strict inclusion. In the cohomology picture this injection is the restriction map $\zeta^0_{n+1}\colon\ H^0(X_{n+1}; \mathcal{O}_{X_{n+1}}(E)) \to H^0(X_n; \mathcal{O}_{X_n}(E))$. Therefore $\varprojlim^{(1)} H^0(X_n; \mathcal{O}_{X_n}(E)) \neq 0$, the action of G on $\varprojlim^{(1)} H^0(X_n; \mathcal{O}_{X_n}(E))$ being trivial. Note also that $H^0(X; \mathcal{O}_X(E)) = \varprojlim H^0(X_n; \mathcal{O}_{X_n}(E)) = \bigcap_n E^{P_n} = 0$.

Now consider $\varprojlim H^k(X_n; \mathcal{O}_{X_n}(E))$. The P_n-module E decomposes as $(E'_n)^* \oplus \widetilde{E_n}$ where $\widetilde{E_n}$ is a direct sum of irreducible P_n-modules not isomorphic to $(E'_n)^*$. Therefore we have a canonical G_n-module injection

$$H^k(X_n; \mathcal{O}_{X_n}((E'_n)^*)) = V(E'_n)^* \hookrightarrow H^k(X_n; \mathcal{O}_{X_n}(E)).$$

Furthermore, the map

$$\zeta^k_{n+1}\colon\ H^k(X_{n+1}; \mathcal{O}_{X_{n+1}}(E)) \to H^k(X_n; \mathcal{O}_{X_n}(E))$$

induces a surjection

$$H^k(X_{n+1}; \mathcal{O}_{X_{n+1}}((E'_{n+1})^*)) \to H^k(X_n; \mathcal{O}_{X_n}((E'_n)^*)).$$

Thus there is a canonical G-module injection

$$V(E')^* = \varprojlim_{n>k} V(E'_n)^* \hookrightarrow \varprojlim H^k(X_n; \mathcal{O}_{X_n}(E)),$$

and, in particular, $H^k(X; \mathcal{O}_X(E)) \neq 0$.

A similar argument shows that $\varprojlim H^i(X_n; \mathcal{O}_{X_n}(E)) \neq 0$ for $1 \leq i < k$. In effect, for any such fixed i the P_n-module E decomposes as $(E^i_n)^* \oplus \widetilde{E^i_n}$, where E^i_n is the irreducible P_n-module with B_n-highest weight $2k\varepsilon_2 + (2k-1)\varepsilon_3 + \cdots + (i+1)\varepsilon_{2k-i+1}$ and where $\widetilde{E^i_n}$ is a direct sum of irreducible P_n-modules not isomorphic to $(E_n^i)^*$. Then, as above, one verifies that there is a G-module injection

$$V(\varinjlim_{n>2k-i+1} E_n)^* \hookrightarrow \varprojlim H^i(X_n; \mathcal{O}_{X_n}(E)).$$

Consequently $H^i(X; \mathcal{O}_X(E)) \neq 0$ for $1 \leq i \leq k$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095
E-mail: dimitrov@math.ucla.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521
E-mail: penkov@math.ucr.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840
E-mail: jawolf@math.berkeley.edu

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