Orthogonal complement

Let $V$ be an inner product space. $\dim V < \infty$.

**Def.** If $W$ is a subspace of $V$, $W^\perp = \{ v \in V \mid (v, w) = 0 \text{ for every } w \in W \}$.

$W^\perp$ is called the orthogonal complement of $W$.

$W^\perp$ consists of all vectors of $V$ that are perpendicular to all vectors in $W$. 
Example, (i) $V = \mathbb{R}^3$ with dot product.

(ii) $W$ - line $l$

$W^\perp$ is the plane $2$ orthogonal to the line $l$.

(iii) $W$ - plane $2$

$W^\perp$ - line $l$ or orthogonal to $2$

(iv) $W = 0 \Rightarrow W^\perp = \mathbb{R}^3$

(iv) $W = \mathbb{R}^3 \Rightarrow W^\perp = 0$
Proposition. (i) $W^+$ is a subspace of $U$

(ii) If $W = \text{span}\{w_1, w_2, \ldots, w_k\}$ then

\[ v \in W^+ \iff (v, w_1) = (v, w_2) = \ldots = (v, w_k) = 0. \]

Proof. (i) $u, v \in W^+$ \implies u + v \in W^+$

For any $w \in W$

\[ (u, w) = 0 \]
\[ (v, w) = 0 \]

\[ u \in W^+, \lambda \in \mathbb{R} \]

\[ \Rightarrow \lambda u \in W^+ \]

For any $w \in W$

\[ (u, w) = 0 \]
\[ (\lambda u, w) = \lambda (u, w) = 0 \]

\[ \Rightarrow \lambda u \in W^+ \]
(\text{(i)}) \quad \forall v \in W^+ \iff (v, v) = (v, w_2) = \ldots = (v, w_k) = 0,
\Rightarrow \quad \forall v \in W^+ \Rightarrow (v, w) = 0 \quad \text{for every } w \in W.

\text{Since } w_1, w_2, \ldots, w_k \in W \Rightarrow (v, w_1) = 0, (v, w_2) = 0, \ldots
\leq \quad \text{If } (v, w_1) = (v, w_2) = \ldots = (v, w_k) = 0
\quad \text{and } w \in W \iff \text{span } \{w_1, w_2, \ldots, w_k\} =
\quad w = c_1 w_1 + c_2 w_2 + \ldots + c_k w_k,
\text{hence } \quad (v, w) = (v, c_1 w_1 + c_2 w_2 + \ldots + c_k w_k) =
\quad = c_1 (v, w_1) + c_2 (v, w_2) + \ldots + c_k (v, w_k) = 0
\quad \Rightarrow \quad \forall v \in W^+ \quad 0 = 0.
\quad \Rightarrow \quad \forall v \in W^+ \quad 0 = 0.
Example 2. Let $A$ be an $m \times n$ matrix. Are these related?

\[
\text{Row}(A) \subseteq \mathbb{R}^n \quad \text{Ker}(A) \subseteq \mathbb{R}^m
\]
\[
\text{Col}(A) = \text{Im}(A) \subseteq \mathbb{R}^m
\]

\[
\text{Ker } A = (\text{Row } A)^\perp
\]

Indeed, $A = \begin{bmatrix} A_1^T & A_2^T & \cdots & A_m^T \end{bmatrix}$, for $x \in \mathbb{R}^n$

\[
A^T x = 0 \iff \begin{cases} A_1^T x = 0, \\ A_2^T x = 0, \\ \vdots \\ A_m^T x = 0 \end{cases}
\]

\[
C = \overrightarrow{\text{span}} \left\{ \overrightarrow{A_1}, \overrightarrow{A_2}, \ldots, \overrightarrow{A_m} \right\} = (\text{Row } A)^\perp
\]
(3) \( W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{F}^5 \)

Find a basis of \( W^+ \),

\( W = \text{Rov A} \)

\( W^+ = \text{ker A} \)

A basis of \( W^+ = \text{ker A} \) is

\[
\begin{bmatrix} -5 \\ -4 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\( \text{free free} \)
Example 4 \( V = \mathbb{R}^\infty = \{ (a_1, a_2, a_3, \ldots ) \mid a_i \neq 0 \text{ for finitely many } a_i \} \)

\[(a_1, a_2, a_3, \ldots ), (b_1, b_2, b_3, \ldots ) = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots \]

inner product in \( \mathbb{R}^\infty \)

\[ W = \text{span} \{ (1, -1, 0, 0, \ldots ), (0, 1, -1, 0, 0, \ldots ), (0, 0, 1, -1, 0, 0, \ldots ), \ldots \} \]

\[ W^\perp = 0 \quad \underbrace{= V^\perp}_{W \subset V, \ W^\perp = 0} \]

If \( v = (c_1, c_2, c_3, c_4, \ldots ) \in W^\perp \Rightarrow \]

\( c_1 = c_2 = c_3 = c_4 = \cdots = 0 \)
\[ \forall v \in W^+ \Rightarrow v^* (c, c, c, c, \ldots) \in R^\infty \Rightarrow c \to 0 \text{ because } v \text{ only has finitely many non-zero entries!} \]

**Theorem**. Let \( V \) be a f.d. inner product space, and \( WC V \) be subspace. Then

(i) \( W \cap W^+ = 0 \)

(ii) \( V = W + W^+ \).

**Proof**. (i) Assume \( x \in W \cap W^+ \). Then \( (x, x) = 0 \Rightarrow x = 0 \).
Thus $v = W^\perp + W$

Remarks

(i) For $W \subset \mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R}$, $w^\perp = 0$ 

(ii) For $W \subset \mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R}$, $v \neq w + w^\perp$, $\text{proj}_W v$ does not always exist!

$v = (1, 0, 0, \ldots)$ has no projection.
Corollary. \( \text{dim } V = \text{dim } W + \text{dim } W^\perp \).

Idea of proof. If \( B_1 \) is a basis of \( V \)
\[ B_1 \cap B_2 = \emptyset \]
\[ B_2 \] is a basis of \( W^\perp \)
\[ \Rightarrow B = B_1 \cup B_2 \] is a basis of \( V \)
\[ \text{dim } V = \# B = \# (B_1 \cup B_2) = \# B_1 + \# B_2 = \text{dim } W + \text{dim } W^\perp. \]

Application. For an \( m \times n \) matrix \( A \)
\[ (\text{Row } A)^\perp = \ker A \subset \mathbb{R}^n \]
\[ \text{dim } \mathbb{R}^n = n = \text{dim } (\text{Row } A)^\perp + \text{dim } (\text{Row } A)^\perp \]
\[ = \text{rk } A + \text{dim } (\ker A)^\perp \]
Proposition \((W^\perp)^\perp = W\), \((V\text{ is } F.d.)\)

Proof. (i) \(W \subset (W^\perp)^\perp\):
If \(w \in W\) \(\Rightarrow\) \(\bigcirc\) for any \(v \in W^\perp\)
\(\Rightarrow\) \(w \in (W^\perp)^\perp\).
(ii) \((W^\perp)^\perp \subset W\):
Let \(x \in (W^\perp)^\perp\) \(\Rightarrow\) \(x \in V\) \(\Rightarrow\) by the theorem.
\[
x = y + 2
\]
\[
y \in W \Rightarrow w^T x \in W^T
\]
\[
(2, 2) = (2, x - y) = (2, x) - (2, y) = 0
\]
\[
\Rightarrow (2, 2) = 0 \Rightarrow \exists z = 0 \Rightarrow x = y \in W
\]
\[
\text{proving } (W^T)^T \subseteq W.
\]
Again, if \( V \) is not f.d., this may not be true: \( W \subset \mathbb{R}^\infty \) from Ex. 4.
\[
W^T \neq 0 \quad (W^T)^T \neq 0^T = V \quad W \subset W^T = V.
\]