Problem Set #13

Due: 10 January 2020 at 3:00 p.m.

1. Calculate \( \frac{D_n}{(n+1)!} \), where \( D_n \) denotes the following \( n \times n \)-determinant:

\[
D_n = \det \begin{bmatrix}
3 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & \cdots & 1 \\
1 & 1 & 5 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & n+2
\end{bmatrix}.
\]

**Hint.** Try some small values of \( n \), make a conjecture, and prove it using the properties of determinants.

2. (a) Let 

\[
D(x) = \det \begin{bmatrix}
f_{11}(x) & f_{12}(x) \\f_{21}(x) & f_{22}(x)
\end{bmatrix},
\]

where \( f_{ij}(x) \) is a function of \( x \). Prove that

\[
D'(x) = \det \begin{bmatrix}
f'_{11}(x) & f'_{12}(x) \\f_{21}(x) & f_{22}(x)
\end{bmatrix} + \det \begin{bmatrix}
f_{11}(x) & f_{12}(x) \\f'_{21}(x) & f'_{22}(x)
\end{bmatrix}.
\]

(b) Let \( g_1(x) \) and \( g_2(x) \) be solutions of the differential equation

\[
y'' + a(x)y' + b(x)y = 0
\]

and let

\[
W(x) = \det \begin{bmatrix}
g_1(x) & g_2(x) \\g'_1(x) & g'_2(x)
\end{bmatrix}.
\]

Show that \( W'(x) = -a(x)W(x) \).
3. Let \( \mathbf{u} = (u_1, u_2, u_3) \), \( \mathbf{v} = (v_1, v_2, v_3) \), and \( \mathbf{w} = (w_1, w_2, w_3) \) be three vectors in \( \mathbb{R}^3 \). Recall that the oriented volume \( V(\mathbf{u}, \mathbf{v}, \mathbf{w}) \) of the parallelepiped spanned by \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) equals \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \). (See p. 48 for the definition of the cross product.)

(a) Assuming the formula \( V(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \), prove that

\[
V(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{bmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
\end{bmatrix}.
\]

(b) If \( A \) is a \( 3 \times 3 \)-matrix, prove that

\[
V(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = \det(A) V(\mathbf{u}, \mathbf{v}, \mathbf{w}).
\]