Problem Set #14
Due: 17 January 2020 at 3:00 p.m.

1. (a) In the set $T := \mathbb{R} \cup \{\infty\}$ define “addition” by

$$v + w := \min(v, w)$$

for all $v, w \in T$. Determine which of the axioms of a vector space that relate to addition only (the first 4 axioms) are satisfied for this operation.

(b) Prove that the set $P := \{x \in \mathbb{R} \mid x > 0\}$, with addition and scalar multiplication defined by

$$v + w := vw, \quad \lambda v := \nu^\lambda$$

for all $v, w \in P$ and all $\lambda \in \mathbb{R}$, is a real vector space.

2. (a) Give an example of a nonempty subset $U$ in $\mathbb{R}^2$ such that $U$ is closed under multiplication by scalars, but $U$ is not a linear subspace of $\mathbb{R}^2$.

(b) Give an example of a nonempty subset $W$ in $\mathbb{R}^2$ such that $W$ is closed under addition, but $W$ is not a linear subspace of $\mathbb{R}^2$.

3. Let $V$ be a vector space and let $W_1$ and $W_2$ be subspaces of $V$.

(a) Prove that $W_1 \cap W_2$ also is a subspace of $V$. Is $W_1 \cup W_2$ always a subspace of $V$?

(b) Let $W = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W$ is a subspace of $V$. This subspace is denoted by $W_1 + W_2$. 