Problem Set #16
Due: 4 February 2019 at noon

1. Let $U$ be a finite dimensional vector space, and let $V$ and $W$ be subspaces of $U$. Recall that $V + W$ and $V \cap W$ are subspaces of $U$, see Problem 3 from Assignment 14. Prove that

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W).$$

*Hint.* Start with a basis of $V \cap W$, complete it to bases of $V$ and $W$ respectively and prove that all vectors obtained in this way form a basis of $V + W$.

2. If $a$ is a real number, define the map

$$T_a : \begin{cases} C(\mathbb{R}, \mathbb{R}) & \rightarrow & C(\mathbb{R}, \mathbb{R}) \\ f(x) & \mapsto & f(x + a). \end{cases}$$

(a) Prove that $T$ is a linear transformation.

(b) Show that $T_a \cdot T_b = T_{a+b}$.

(c) Find Ker$T_a$ and Im$T_a$.

(d) Let $W = \text{span}\{x \sin x, x \cos x, \sin x, \cos x\}$. Show that $T_a$ is a transformation which sends vectors in $W$ into vectors in $W$. In other words, the formula above also defines a linear transformation $S_a : W \rightarrow W$.

3. Recall that Mat$_{n \times n}(\mathbb{F})$ denotes the vector space of $n \times n$–matrices with entries in $\mathbb{F}$, define $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$ by $T(A) = A - A^\top$. Show that $T$ is a linear transformation and find its kernel and image.