1. If $a$ is a real number, define the map

$$T_a : \begin{cases} C(\mathbb{R}, \mathbb{R}) &\rightarrow C(\mathbb{R}, \mathbb{R}) \\ f(x) &\mapsto f(x + a). \end{cases}$$

(a) Prove that $T_a$ is a linear transformation.

(b) Show that $T_a \circ T_b = T_{a+b}$.

(c) Let $W = \text{span}\{x \sin x, x \cos x, \sin x, \cos x\}$. Show that $T_a$ is a transformation which sends vectors in $W$ into vectors in $W$. In other words, the formula above also defines a linear transformation $S_a : W \rightarrow W$.

2. Recall that $\text{Mat}_{n \times n}(\mathbb{F})$ denotes the vector space of $n \times n$–matrices with entries in $\mathbb{F}$, define $T : \mathbb{M}_n \rightarrow \mathbb{M}_n$ by $T(A) = A - A^T$. Show that $T$ is a linear transformation and find its kernel and image.

3. Let $U$ be a finite dimensional vector space and let $T : U \rightarrow U$ be a linear transformation. Recall that the rank of $T$ is defined as the dimension of $\text{Im}(T)$.

(a) Prove that $\text{Im}(T^2) \subset \text{Im}(T)$ and $\text{Ker}(T) \subset \text{Ker}(T^2)$.

(b) If $\text{rk} T = \text{rk} T^2$, prove that $\text{Im}(T) = \text{Im}(T^2)$ and $\text{Ker}(T) = \text{Ker}(T^2)$.

(c) If $\text{rk} T = \text{rk} T^2$, prove that $\text{Im}(T) \cap \text{Ker}(T) = 0$ and $U = \text{Im}(T) + \text{Ker}(T)$. 