1. For a matrix $A$ denote the characteristic polynomial of $A$ by $\xi_A(t)$, i.e.

$$\xi_A(t) = \det(A - tI),$$

where $I$ is the identity matrix.

(a) Prove that $\xi_A^2(t^2) = \xi_A(t)\xi_A(-t)$.

(b) Given that $\xi_A(t) = t^4 + t + 1$, find $\xi_A^2(t)$.

2. Let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ be three vectors in $\mathbb{R}^3$. Recall that the oriented volume $V(u, v, w)$ of the parallelepiped spanned by $u, v, w$ equals $(u \times v) \cdot w$. (See p. 48 for the definition of the cross product.)

(a) Assuming the formula $V(u, v, w) = (u \times v) \cdot w$, prove that

$$V(u, v, w) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

(b) If $A$ is a $3 \times 3$–matrix, prove that

$$V(Au, Av, Aw) = \det(A)V(u, v, w).$$

Remark. The formula from (b) shows that $\det(A)$ measures how the linear transformation with matrix $A$ changes the volume of a region in $\mathbb{R}^3$.

3. Let $T : U \to U$ be a linear transformation and let $\mathcal{B}$ be a basis of $U$. Define the determinant $\det(T)$ of $T$ as

$$\det(T) = \det([T]_{\mathcal{B}}).$$

Show that $\det(T)$ is well–defined, i.e. that it does not depend on the choice of the basis $\mathcal{B}$.

Prove that $T$ is invertible if and only if $\det(T) \neq 0$. If $T$ is invertible, show that

$$\det(T^{-1}) = \frac{1}{\det(T)}.$$