Problem Set #24
Due: 8 April 2019

1. (a) Prove that
\[ \langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) \]
defines an inner product in the space \( \mathbb{R}[x]_{\leq 2} \) of polynomials of degree at most 2. Is \( \langle p(x), q(x) \rangle \) an inner product in the space \( \mathbb{R}[x]_{\leq 3} \)?

(b) Prove that
\[ \langle A, B \rangle = -\text{tr}(AB) \]
defines an inner product on the space of skew–symmetric \( n \times n \)–matrices.

2. (a) Let \( \langle \mathbf{u}, \mathbf{v} \rangle \) be an inner product in \( \mathbb{R}^n \). Prove that there exists a symmetric \( n \times n \)–matrix \( A \) such that
\[ \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}. \]

(b) If \( n = 2 \), prove that
\[ a_{11} > 0 \quad \text{and} \quad \det(A) > 0, \]
where \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) is the matrix from (a).

3. Let \( \langle \mathbf{u}, \mathbf{v} \rangle \) be an inner product in a real vector space \( U \). Given \( k \) vectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) in \( U \), set \( a_{ij} := \langle \mathbf{u}_i, \mathbf{u}_j \rangle \).

(a) Prove that the matrix \( A = (a_{ij}) \) is invertible if and only if \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) are linearly independent.

(b) Prove that, for any scalars \( x_1, x_2, \ldots, x_k \in \mathbb{R} \), we have
\[ \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} x_i x_j \geq 0. \]