Solutions #6

1. Let \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) be three vectors in \( \mathbb{R}^3 \). Prove that \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 \) if and only if \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent.

Solution. We need to prove two things:

(a) If \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent, then \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 \).

(b) If \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 \), then \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent.

Below we prove each of these statements.

(a) Let \( \vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} \). Since \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent, the equation

\[
x \vec{v}_1 + y \vec{v}_2 + z \vec{v}_3 = \vec{0}
\]

has the unique solution \( x = y = z = 0 \). This equation is equivalent to the linear system

\[
\begin{cases}
a_1 x + a_2 y + a_3 z = 0 \\
b_1 x + b_2 y + b_3 z = 0 \\
c_1 x + c_2 y + c_3 z = 0
\end{cases}
\]

Thus we obtain that the homogeneous system above has the unique solution \( x = y = z = 0 \) which implies that the RREF of the coefficient matrix \( A \) has no free variables. In particular, \( \text{rk}A = 3 \), where \( A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \) is the coefficient matrix of the system above.

To prove that \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 \), we need to show that every vector \( \vec{u} \in \mathbb{R}^3 \) belongs to \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \). In other words, we need to show that the equation

\[
x \vec{v}_1 + y \vec{v}_2 + z \vec{v}_3 = \vec{u}
\]

has a solution. Let \( \vec{u} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \). The equation above becomes the linear system

\[
\begin{cases}
a_1 x + a_2 y + a_3 z = \alpha \\
b_1 x + b_2 y + b_3 z = \beta \\
c_1 x + c_2 y + c_3 z = \gamma
\end{cases}
\]

To determine whether this system has a solution, we consider its augmented matrix which of the form \( [A | \vec{u}] \), i.e., it is the \( 3 \times 4 \)-matrix whose first three columns form the matrix \( A \) and the last column is made of the entries of the vector \( \vec{u} \). Let’s consider the rank of \( [A | \vec{u}] \).

On one hand, \( \text{rk}[A | \vec{u}] \geq \text{rk}A = 3 \) (Why?) and, on the other hand, \( \text{rk}[A | \vec{u}] \leq 3 \) since \( [A | \vec{u}] \) has only 3 rows. Hence \( \text{rk}[A | \vec{u}] = 3 \). In particular \( \text{rk}[A | \vec{u}] = \text{rk}A \), which implies that the system above has a (unique) solution. This proves that \( \vec{u} \in \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \) and since \( \vec{u} \) was an arbitrary vector in \( \mathbb{R}^3 \), we conclude that \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 \).
(b) We can prove this direction by an argument similar to the one we used in (a) above, however we can also argue in a more geometric way. 
Let span(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3 and assume, to the contrary, that \vec{v}_1, \vec{v}_2, \vec{v}_3 are linearly dependent. This means that one of the three vectors is a linear combination of the other two. Say, \vec{v}_3 is a linear combination of \vec{v}_1 and \vec{v}_2. Thus \vec{v}_3 \in \text{span}(\vec{v}_1, \vec{v}_2). We also know that \vec{v}_1 \in \text{span}(\vec{v}_1, \vec{v}_2) and \vec{v}_2 \in \text{span}(\vec{v}_1, \vec{v}_2) (Why?). Problem 3 from HW 5 implies that 
\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \subset \text{span}(\vec{v}_1, \vec{v}_2).

By assumption \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3, while \text{span}(\vec{v}_1, \vec{v}_2) is at most a plane in 3-space (it may also be a line or just a point). This is a contradiction because the whole 3-space cannot be contained in a plane. This contradiction proves that \vec{v}_1, \vec{v}_2, \vec{v}_3 are linearly independent.

\[ \square \]

2. Find all values of \( k \) for which the vectors

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \end{bmatrix}
\]

are linearly dependent.

Solution. The vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \end{bmatrix}
\]

are linearly dependent if the equation

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = \vec{0} \]

has solutions other than the trivial solution \( c_1 = c_2 = c_3 = c_4 = 0 \). This equation is equivalent to the linear system

\[
\begin{cases}
  c_1 + c_2 + c_3 + c_4 = 0 \\
  c_2 - c_3 + kc_4 = 0 \\
  c_2 + c_3 + k^2c_4 = 0 \\
  c_2 - c_3 + k^3c_4 = 0.
\end{cases}
\]

In particular, the four vectors are linearly dependent exactly when the system above has free variables. Row-reduction gives

\[
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & k & 0 \\ 0 & 1 & 1 & k^2 & 0 \\ 0 & 1 & -1 & k^3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & k & 0 \\ 0 & 0 & 1 & k^2 - k & 0 \\ 0 & 0 & 0 & k^3 - k & 0 \end{bmatrix}.
\]
This shows that

\[ \text{rk}A = \begin{cases} 4 & \text{if } k^3 - k \neq 0 \\ 3 & \text{if } k^3 - k = 0. \end{cases} \]

In other words, the linear system has free variable if and only if \( k^3 - k = 0 \), which implies that the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) are linearly dependent for \( k = -1, 0, 1 \). \( \square \)

3. Let \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) be linearly independent vectors in \( \mathbb{R}^m \).

(a) Show that the vectors \( \vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \ldots, \vec{v}_{n-1} - \vec{v}_n, \vec{v}_n \) are also linearly independent.

(b) Suppose that, for some vector \( \vec{w} \in \mathbb{R}^m \), the vectors \( \vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \ldots, \vec{v}_n + \vec{w} \) are linearly dependent. Prove that \( \vec{w} \in \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \).

\textbf{Solution.}

(a) Suppose we have a linear relation among \( \vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \ldots, \vec{v}_{n-1} - \vec{v}_n, \vec{v}_n \), i.e., there are \( c_1, \ldots, c_n \in \mathbb{R} \) such that \( c_1(\vec{v}_1 - \vec{v}_2) + c_2(\vec{v}_2 - \vec{v}_3) + \cdots + c_{n-1}(\vec{v}_{n-1} - \vec{v}_n) + c_n \vec{v}_n = \vec{0} \). Re-arranging the left-side of the equation yields

\[ c_1\vec{v}_1 + (c_2 - c_1)\vec{v}_2 + (c_3 - c_2)\vec{v}_3 + \cdots + (c_{n-1} - c_{n-2})\vec{v}_{n-1} + (c_n - c_{n-1})\vec{v}_n = \vec{0}. \]

Since \( \vec{v}_1, \ldots, \vec{v}_n \) are linearly independent in \( \mathbb{R}^n \), we conclude that

\[ c_1 = 0 \quad c_2 - c_1 = 0 \quad c_3 - c_2 = 0 \quad \cdots \quad c_{n-1} - c_{n-2} = 0 \quad c_n - c_{n-1} = 0. \]

Solving this system gives \( 0 = c_1 = c_2 = c_3 = \cdots = c_{n-1} = c_n \). Therefore, the linear relation is trivial and the vectors \( \vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \ldots, \vec{v}_{n-1} - \vec{v}_n, \vec{v}_n \) are linearly independent.

(b) Since \( \vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \ldots, \vec{v}_n + \vec{w} \) are linearly dependent, there exist scalars \( c_1, c_2, \ldots, c_n \), not all equal to zero, such that

\[ c_1(\vec{v}_1 + \vec{w}) + c_2(\vec{v}_2 + \vec{w}) + \cdots + c_n(\vec{v}_n + \vec{w}) = \vec{0}. \]

Rearranging the terms above we obtain

\[ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + (c_1 + c_2 + \cdots + c_n)\vec{w} = \vec{0}, \]

or

\[ (c_1 + c_2 + \cdots + c_n)\vec{w} = -c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_n\vec{v}_n. \]

We now consider two cases for \( c_1 + c_2 + \cdots + c_n \).

(i) If \( c_1 + c_2 + \cdots + c_n \neq 0 \), we can divide by it to obtain

\[ \vec{w} = -\frac{c_1}{c_1 + c_2 + \cdots + c_n} \vec{v}_1 - \frac{c_2}{c_1 + c_2 + \cdots + c_n} \vec{v}_2 - \cdots - \frac{c_n}{c_1 + c_2 + \cdots + c_n} \vec{v}_n = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n, \]

where \( b_1 = -\frac{c_1}{c_1 + c_2 + \cdots + c_n}, b_2 = -\frac{c_2}{c_1 + c_2 + \cdots + c_n}, \ldots, b_n = -\frac{c_n}{c_1 + c_2 + \cdots + c_n} \). This proves that \( \vec{w} \in \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \).
(ii) If \( c_1 + c_2 + \ldots + c_n = 0 \), then the equation before the cases implies that
\[
\vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0},
\]
which together with the linear independence of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) implies that \( c_1 = c_2 = \ldots = c_n = 0 \) which contradicts our assumption. This proves that Case (ii) is impossible.
\( \square \)