1. Let
\[ A(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \]

Prove that
(a) \( A(\theta)A(\varphi) = A(\theta + \varphi); \)
(b) \( (A(\theta))^n = A(n\theta) \) for every positive integer \( n. \)

Solution.
(a) We calculate
\[
A(\theta)A(\varphi) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}
= \begin{bmatrix} \cos \theta \cos \varphi - \sin \theta \sin \varphi & \cos \theta \sin \varphi + \sin \theta \cos \varphi \\ -\sin \theta \cos \varphi - \cos \theta \sin \varphi & -\sin \theta \sin \varphi + \cos \theta \cos \varphi \end{bmatrix}
= \begin{bmatrix} \cos(\theta + \varphi) & \sin(\theta + \varphi) \\ -\sin(\theta + \varphi) & \cos(\theta + \varphi) \end{bmatrix}.
\]

(b) We prove this statement by induction on \( n. \) For \( n = 1 \) the identity is obvious. Assume we have proved it for \( n, \) i.e., we have proved that \( (A(\theta))^n = A(n\theta). \) Then, using (a), we can calculate
\[
(A(\theta))^{n+1} = (A(\theta))^nA(\theta) = A(n\theta)A(\theta) = A(n\theta + \varphi) = A((n+1)\theta).
\]
This completes the proof of the induction step and hence of the solution.

2. A square matrix \( M \) is skew-symmetric if \( M^\top = -M. \)

(a) Prove that any square matrix \( M \) can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
(b) Illustrate part (a) for the matrix $M = \begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix}$.

**Solution.**

(a) Let $M$ be any square matrix. The properties of the transpose yield

$$
\left( \frac{1}{2}(M + M^T) \right)^T = \frac{1}{2}M^T + \frac{1}{2}(M^T)^T = \frac{1}{2}M^T + \frac{1}{2}M = \frac{1}{2}(M + M^T)
$$

$$
\left( \frac{1}{2}(M - M^T) \right)^T = \frac{1}{2}M^T - \frac{1}{2}(M^T)^T = \frac{1}{2}M^T - \frac{1}{2}M = -\left( \frac{1}{2}(M - M^T) \right).
$$

Hence, $\frac{1}{2}(M + M^T)$ is symmetric and $\frac{1}{2}(M - M^T)$ is skew-symmetric. Therefore, the equation $M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$ expresses $M$ as the sum of a symmetric matrix and a skew-symmetric matrix.

(b) We have

$$
\begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 11 & 14 & 16 \\ 5 & 2 & 12 & 15 \\ 8 & 6 & 3 & 13 \\ 10 & 9 & 7 & 4 \end{bmatrix} \right) + \frac{1}{2} \left( \begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix} \right) 
$$

$$
= \begin{bmatrix} 1 & 8 & 11 & 13 \\ 8 & 2 & 9 & 12 \\ 11 & 9 & 3 & 10 \\ 13 & 12 & 10 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -3 & -3 \\ 3 & 0 & -3 & -3 \\ 3 & 3 & 0 & -3 \\ 3 & 3 & 3 & 0 \end{bmatrix}.
$$

\[ \square \]

3. The **trace** of an $(n \times n)$-matrix $A = [a_{i,j}]$ is the sum of its diagonal entries:

$$
\text{tr}(A) := a_{1,1} + a_{2,2} + \cdots + a_{n,n}.
$$

(a) Prove that the trace is linear; in other words, show that, for any $(n \times n)$-matrices $A$, $B$ and any scalar $c$, we have $\text{tr}(cA + B) = c \text{tr}(A) + \text{tr}(B)$.

(b) If $A$ and $B$ are $(n \times n)$-matrices, then prove that $\text{tr}(AB) = \text{tr}(BA)$. 
(c) For \( n \geq 1 \), show that \( XY - YX = I_n \) has no solutions for \((n \times n)\)-matrices \( X \) and \( Y \). Here \( I_n \) stands for the identity matrix of size \( n \times n \).

Solution. Let \( A = [a_{i,j}] \) and \( B = [b_{i,j}] \).

(a) We have
\[
c \text{tr}(A) + \text{tr}(B) = c(a_{1,1} + a_{2,2} + \cdots + a_{n,n}) + (b_{1,1} + b_{2,2} + \cdots + b_{n,n})
= (ca_{1,1} + b_{1,1}) + (ca_{2,2} + b_{2,2}) + \cdots + (ca_{n,n} + b_{n,n})
= \text{tr} \left( [ca_{i,j} + b_{i,j}] \right) = \text{tr}(cA + B).
\]

(b) The definition of matrix multiplication implies that the \((i, j)\)-entry of \( AB \) is
\[
a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j},
\]
and the \((i, j)\)-entry of \( BA \) is
\[
b_{i,1}a_{1,j} + b_{i,2}a_{2,j} + \cdots + b_{i,n}a_{n,j} = \sum_{k=1}^{n} b_{i,k}a_{k,j}.
\]
Hence, we obtain
\[
\text{tr}(AB) = \sum_{\ell=1}^{n} \sum_{k=1}^{n} a_{\ell,k}b_{k,\ell} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} b_{k,\ell}a_{\ell,k} = \text{tr}(BA)
\]
by using the commutativity of multiplication of numbers and changing the ordering of summation.

(c) The identity matrix has 1 in each diagonal entry, so we have \( \text{tr}(I_n) = n \). On the other hand, parts (a) and (b) imply that \( \text{tr}(XY - YX) = \text{tr}(XY) - \text{tr}(YX) = \text{tr}(XY) - \text{tr}(XY) = 0 \). Since taking the trace of both sides of the equation \( XY - YX = I \) produces a contradiction, the equation \( XY - YX = I \) has no solutions for \((n \times n)\)-matrices \( X \) and \( Y \). \( \Box \)