Solutions #14

1. 

(a) In the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ define “addition” by

$$v + w := \min(v, w)$$

for all $v, w \in \mathbb{T}$. Determine which of the axioms of a vector space that relate to addition only (the first 4 axioms) are satisfied for this operation.

(b) Prove that the set $\mathbb{P} := \{x \in \mathbb{R} \mid x > 0\}$, with addition and scalar multiplication defined by

$$v + w := vw, \quad \lambda v := v^\lambda$$

for all $v, w \in \mathbb{P}$ and all $\lambda \in \mathbb{R}$, is a real vector space.

Solution.  

(a) We have

$$v + w = \min(v, w) = \min(w, v) = w + v$$

$$(u + v) + w = \min(\min(u, v), w) = \min(u, v, w) = \min(u, \min(v, w)) = u + (v + w)$$

$$v + \infty = \min(v, \infty) = v$$

This shows that the first 3 axioms are satisfied with $\infty$ as the “zero vector” in $\mathbb{T}$. In contrast,

$$v + w = \min(v, w) = \infty$$

if and only if both $v$ and $w$ equal $\infty$. This shows that only $\infty$ has an “opposite vector”.

(b) We verify all of the defining properties as follows. For all $u, v, w \in \mathbb{P}$ and all $c, d \in \mathbb{R}$, we have

$$v + w = vw = wv = w + v$$

$$v + (v^{-1}) = vv^{-1} = 1$$

$$(u + v) + w = (uv)w = u(vw) = u + (v + w)$$

$$1v = v^1 = v$$

$$(cd)v = v^{cd} = (v^d)^c = c(dv)$$

$$c(v + w) = (vw)^c = v^cw^c = (cv) + (cw)$$

$$v + 1 = v1 = v$$

$$(c + d)v = v^{c+d} = v^c v^d = (cv) + (dv).$$

In particular, the number $1 \in \mathbb{P}$ is the additive identity. \qed
2. 

(a) Give an example of a nonempty subset $U$ in $\mathbb{R}^2$ such that $U$ is closed under multiplication by scalars, but $U$ is not a linear subspace of $\mathbb{R}^2$.

(b) Give an example of a nonempty subset $W$ in $\mathbb{R}^2$ such that $W$ is closed under addition, but $W$ is not a linear subspace of $\mathbb{R}^2$.

Solution. (a) Consider $U = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. In other words, $U$ is the union of the $x$-axis and the $y$-axis. Since each axis is a line through the origin, each axis is a linear subspace. Hence, each axis is closed under scalar multiplication, so $U$ is also closed under scalar multiplication. However, the subset $U$ is not closed under addition; for example, we have

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in U \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U.
\]

Therefore, $U$ is not a subspace of $\mathbb{R}^2$.

(b) Let $W = \{(x, y) \mid x > 0, y = 0\}$. It is clearly closed under addition but, since it does not contain the zero vector, it is not a subspace of $\mathbb{R}^2$. □

3. Let $V$ be a vector space and let $W_1$ and $W_2$ be subspaces of $V$.

(a) Prove that $W_1 \cap W_2$ also is a subspace of $V$. Is $W_1 \cup W_2$ always a subspace of $V$?

(b) Let $W = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W$ is a subspace of $V$. This subspace is denoted by $W_1 + W_2$.

Solution. (a) Let $u, v \in W_1 \cap W_2$ and let $\lambda \in \mathbb{F}$. Then $u, v \in W_1$ and hence $u + v \in W_1$ and $\lambda u \in W_1$. Similarly, $u, v \in W_2$ and hence $u + v \in W_2$ and $\lambda u \in W_2$. These imply that $u + v \in W_1 \cap W_2$ and $\lambda u \in W_1 \cap W_2$, which completes the proof that $W_1 \cap W_2$ is a subspace of $V$.

$W_1 \cup W_2$ is not necessarily a subspace of $V$. For example, take $V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$, $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$, and $W_2 = \{(0, y) \mid y \in \mathbb{R}\}$. Clearly $W_1$ and $W_2$ are subspaces of $V$ (these are the coordinate axes) but $W_1 \cup W_2$ is not a subspace, since $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

(b) Let $W = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W$ is a subspace of $V$. This subspace is denoted by $W_1 + W_2$. 

Solution. (a) Let $u, v \in W_1 \cap W_2$ and let $\lambda \in \mathbb{F}$. Then $u, v \in W_1$ and hence $u + v \in W_1$ and $\lambda u \in W_1$. Similarly, $u, v \in W_2$ and hence $u + v \in W_2$ and $\lambda u \in W_2$. These imply that $u + v \in W_1 \cap W_2$ and $\lambda u \in W_1 \cap W_2$, which completes the proof that $W_1 \cap W_2$ is a subspace of $V$.

$W_1 \cup W_2$ is not necessarily a subspace of $V$. For example, take $V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$, $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$, and $W_2 = \{(0, y) \mid y \in \mathbb{R}\}$. Clearly $W_1$ and $W_2$ are subspaces of $V$ (these are the coordinate axes) but $W_1 \cup W_2$ is not a subspace, since $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$. 


Remark. \( W_1 \cup W_2 \) is a subspace if and only if \( W_1 \subset W_2 \) or \( W_2 \subset W_1 \). Indeed, if \( W_1 \subset W_2 \), then \( W_1 \cup W_2 = W_2 \) is a subspace of \( V \). Conversely, assume that \( W_1 \cup W_2 \) is a subspace but neither one contains the other one. Take \( w_1 \in W_1 \) which is not contained in \( W_2 \) and \( w_2 \in W_2 \) which is not contained in \( W_1 \). Then \( w = w_1 + w_2 \in W_1 \cup W_2 \). This implies that \( w \in W_1 \) or \( w \in W_2 \). If, for example, \( w \in W_1 \), then \( w_2 = w - w_1 \in W_1 \), which contradicts the choice of \( w_2 \). This contradiction shows that if \( W_1 \cup W_2 \) is a subspace of \( V \), then one of them contains the other one.

(b) Let \( u, v \in W_1 + W_2 \). Then \( u = w'_1 + w'_2 \) and \( v = w''_1 + w''_2 \) for some \( w'_1, w''_1 \in W_1 \) and \( w'_2, w''_2 \in W_2 \). Then \( u + v = (w'_1 + w'_2) + (w''_1 + w''_2) = (w'_1 + w''_1) + (w'_2 + w''_2) \in W_1 + W_2 \). Similarly, \( \lambda u = \lambda (w'_1 + w'_2) = (\lambda w'_1) + (\lambda w'_2) \in W_1 + W_2 \), which completes the proof that \( W_1 + W_2 \) is a subspace of \( V \). \( \square \)