Solutions #16

1. Let $U$ be a finite dimensional vector space, and let $V$ and $W$ be subspaces of $U$. Recall that $V + W$ and $V \cap W$ are subspaces of $U$, see Problem 3 from Assignment 14. Prove that

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W).$$

Solution. Let

$$\{u_1, u_2, \ldots, u_k\}$$

be a basis of $V \cap W$. Since $V \cap W$ is a subspace of both $V$ and $W$, we can complete $\{u_1, u_2, \ldots, u_k\}$ to a basis

$$\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s\}$$

of $V$ and to a basis

$$\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_l\}$$

of $W$. If we can prove that

$$\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\}$$

is a basis of $V + W$, we will be done. Indeed, assuming that, we would have

$$\dim V = k + s, \quad \dim W = k + l, \quad \dim(V + W) = k + s + l, \quad \dim(V \cap W) = k,$$

which implies the required identity.

So, to complete the proof, we will show that $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\}$ are linearly independent and span $V + W$.

To prove that they are linearly independent, assume that

$$a_1u_1 + a_2u_2 + \ldots + a_ku_k + b_1v_1 + b_2v_2 + \ldots + b_s v_s + c_1w_1 + c_2w_2 + \ldots + c_lw_l = 0.$$ 

Rearranging the terms above we obtain

$$a_1u_1 + a_2u_2 + \ldots + a_ku_k + b_1v_1 + b_2v_2 + \ldots + b_s v_s = -(c_1w_1 + c_2w_2 + \ldots + c_lw_l).$$

Let $u$ denote this vector, i.e.,

$$u = a_1u_1 + a_2u_2 + \ldots + a_ku_k + b_1v_1 + b_2v_2 + \ldots + b_s v_s = -(c_1w_1 + c_2w_2 + \ldots + c_lw_l).$$
respectively, we can write
\[ w \in u \]
Now we prove that \( u \)
the proof that \( u \)
Now equation (**) shows that
\[ u = f_1u_1 + f_2u_2 + \ldots + f_ku_k \]
for some scalars \( f_1, f_2, \ldots, f_k \). Now combining two of the expressions for \( u \) above we get
\[ u = f_1u_1 + f_2u_2 + \ldots + f_ku_k = - (c_1w_1 + c_2w_2 + \ldots + c_lw_l), \]
which, after some rearrangement gives
\[ f_1u_1 + f_2u_2 + \ldots + f_ku_k + c_1w_1 + c_2w_2 + \ldots + c_lw_l = 0. \]
Recalling that \( \{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_l\} \) is a basis of \( W \) (and hence linearly independent), we conclude that
\[ f_1 = f_2 = \ldots = f_k = c_1 = c_2 = \ldots = c_l = 0. \]
Now equation (**) shows that \( u = 0 \), which in turn, taking (*) into account implies that
\[ a_1 = a_2 = \ldots = a_k = b_1 = b_2 = \ldots = b_s = 0. \]
Since we established that
\[ a_1 = a_2 = \ldots = a_k = b_1 = b_2 = \ldots = b_s = c_1 = c_2 = \ldots = c_l = 0, \]
the proof that \( u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l \) are linearly independent is complete.
Now we prove that \( u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l \) span \( V + W \). It is clear that
\[ \text{span}\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\} \subseteq V + W. \]
(Why?) To prove the converse inclusion, consider \( u \in V + W \). Then \( u = v + w \) for some \( v \in V \) and \( w \in W \). Since \( \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s\} \) and \( \{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_l\} \) are bases of \( V \) and \( W \) respectively, we can write
\[ v = a'_1u_1 + a'_2u_2 + \ldots + a'_ku_k + b_1v_1 + b_2v_2 + \ldots + b_sv_s \]
and

\[ w = a''_1u_1 + a''_2u_2 + \ldots + a''_ku_k + c_1w_1 + c_2w_2 + \ldots + c_lw_l. \]

Hence

\[
\begin{align*}
    u &= v + w = (a'_1u_1 + \ldots + a'_ku_k + b_1v_1 + \ldots + b_sv_s) + (a''_1u_1 + \ldots + a''_ku_k + c_1w_1 + \ldots + c_lw_l) \\
    &= (a'_1 + a''_1)u_1 + \ldots + (a'_k + a''_k)u_k + b_1v_1 + \ldots + b_sv_s + c_1w_1 + \ldots + c_lw_l.
\end{align*}
\]

The last equation shows that

\[ u \in \text{span}\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\} \]

and, since \( u \) was assumed to be an arbitrary vector in \( V + W \), that

\[ V + W \subseteq \text{span}\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\}. \]

This completes the proof that \( \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_l\} \) spans \( V + W \) and also the solution of the problem. \( \square \)

2. If \( a \) is a real number, define the map

\[ T_a : \begin{cases} 
    C(\mathbb{R}, \mathbb{R}) & \rightarrow & C(\mathbb{R}, \mathbb{R}) \\
    f(x) & \mapsto & f(x + a). 
\end{cases} \]

(a) Prove that \( T \) is a linear transformation.

(b) Show that \( T_a \cdot T_b = T_{a+b} \).

(c) Find \( \text{Ker}T_a \) and \( \text{Im}T_a \).

(d) Let \( W = \text{span}\{x\sin x, x\cos x, \sin x, \cos x\} \). Show that \( T_a \) is a transformation which sends vectors in \( W \) into vectors in \( W \). In other words, the formula above also defines a linear transformation \( S_a : W \rightarrow W \).

Solution. (a) Since, for every \( f, g \in C(\mathbb{R}, \mathbb{R}) \) and every \( x \in \mathbb{R} \), we have

\[ (T_a(f + g))(x) = (f + g)(x + a) = f(x + a) + g(x + a) = T_a(f)(x) + T_a(g)(x) = (T_a(f) + T_a(g))(x) \]

we conclude that \( T_a(f + g) = T_a(f) + T_a(g) \) for every \( f, g \in C(\mathbb{R}, \mathbb{R}) \).
Similarly, for every $f \in C(\mathbb{R}, \mathbb{R})$, every $\lambda \in \mathbb{R}$, and every $x \in \mathbb{R}$, the identity

$$T_a(\lambda f)(x) = (\lambda f)(x + a) = \lambda f(x + a) = \lambda T_a(f)(x)$$

implies that $T_a(\lambda f) = \lambda T_a(f)$ for every $f \in C(\mathbb{R}, \mathbb{R})$ and every $\lambda \in \mathbb{R}$, completing the proof that $T_a$ is a linear transformation.

(b) Computing

$$(T_a \cdot T_b)(f(x)) = T_a(T_b(f(x))) = T_a(f(x + b)) = f((x + b) + a) = f(a + b) = T_{a+b}(f(x)),$$

we see that $T_a \cdot T_b = T_{a+b}$.

(c) Let $f \in C(\mathbb{R}, \mathbb{R})$. Then $T_a(f) = 0$ if, for every $x \in \mathbb{R}$, $f(x + a) = 0$. The last is equivalent to $f(y) = 0$ for every $y \in \mathbb{R}$. Thus, $\text{Ker} T = 0$.

Let $g \in C(\mathbb{R}, \mathbb{R})$ and let $f(x) := g(x - a)$. Then $T_a(f(x)) = f(x + a) = g((x + a) - a) = g(x)$, i.e., $T_a(f) = g$, proving that $g \in \text{Im} T$. Since $g$ is an arbitrary element of $C(\mathbb{R}, \mathbb{R})$, we conclude that $\text{Im} T = C(\mathbb{R}, \mathbb{R})$.

(d) Set $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$ and compute

$$T_a(f_1(x)) = f_1(x + a) = (x + a) \sin(x + a) = (x + a)(\cos a \sin x + \sin a \cos x)$$

$$= \cos ax \sin x + \sin ax \cos x + a \cos ax \sin x + a \sin ax \cos x$$

$$= \cos af_1(x) + \sin af_2(x) + a \cos af_3(x) + a \sin af_4(x);$$

$$T_a(f_2(x)) = f_2(x + a) = (x + a) \cos(x + a) = (x + a)(\cos a \cos x - \sin a \sin x)$$

$$= -\sin ax \sin x + \cos ax \cos x - a \sin ax \sin x + a \cos ax \cos x$$

$$= -\sin af_1(x) + \cos af_2(x) - a \sin af_3(x) + a \cos af_4(x);$$

$$T_a(f_3(x)) = f_3(x + a) = \sin(x + a) = \cos a \sin x + \sin a \cos x = \cos af_3(x) + \sin af_4(x);$$

$$T_a(f_4(x)) = f_4(x + a) = \cos(x + a) = \cos a \cos x - \sin a \sin x = -\sin af_3(x) + \cos af_4(x).$$

If $f \in W$, then $f = c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4$ and

$$T_a(f) = c_1 T_a(f_1) + c_2 T_a(f_2) + c_3 T_a(f_3) + c_4 T_a(f_4) \in \text{span}\{f_1, f_2, f_3, f_4\} = W. \square$$
3. Recall that $M_n(F)$ denotes the vector space of $n \times n$–matrices with entries in $F$, define $T : M_n \to M_n$ by $T(A) = A - A^T$. Show that $T$ is a linear transformation and find its kernel and image.

Solution. If $A, B \in M_n$, then we have

$$T(A + B) = (A + B) - (A + B)^T = A - A^T + B - B^T = (A - A^T) + (B - B^T) = T(A) + T(B).$$

Similarly, if $\lambda \in F$ and $A \in M_n$, we have

$$T(\lambda A) = \lambda A - (\lambda A)^T = \lambda A - \lambda A^T = \lambda (A - A^T) = \lambda T(A),$$

which proves that $T$ is a linear transformation.

For the rest of the solution denote by $\mathcal{S}_n$ the space of symmetric $n \times n$–matrices, and by $\mathcal{A}_n$ — the space of skew–symmetric $n \times n$–matrices.

A matrix $A$ belongs to the kernel of $T$ if and only if $T(A) = A - A^T = 0$, which is equivalent to $A = A^T$. This proves that the kernel of $T$ is the space of symmetric $n \times n$–matrices $\mathcal{S}_n$.

If a matrix $B$ belongs to the image of $T$, then $B = T(A) = A - A^T$ for some matrix $A$. We now compute

$$B^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -B,$$

which implies that $\text{Im} \, T \subset \mathcal{A}_n$.

It remains to show that actually $\text{Im} \, T = \mathcal{A}_n$. This can be done in two ways.

The first one is to observe that if $B \in \mathcal{A}_n$, i.e. if $B = -B^T$, then

$$T(B) = B - B^T = B - (-B) = 2B,$$

i.e. $B = \frac{1}{2} T(B) = T(\frac{1}{2} B)$, which implies that $B \in \text{Im} \, T$, which shows that $\mathcal{A}_n \subset \text{Im} \, T$.

The second one is to use the Rank Theorem:

$$\dim \, M_n = \dim \, \text{Ker} \, T + \dim \, \text{Im} \, T$$

and the fact that $\dim \, M_m = n^2$ and $\dim \, \text{Ker} \, T = \dim \, \mathcal{S}_n = \frac{n(n+1)}{2}$. (The latter fact is similar to Problem 1 from the previous homework.). Then the Rank Theorem gives

$$\dim \, \text{Im} \, T = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$
Since $\text{Im} T \subset \mathcal{A}_n$ and $\dim \text{Im} T = \dim \mathcal{A}_n = \frac{n(n-1)}{2}$, we conclude that $\text{Im} T = \mathcal{A}_n$. \qed