1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define the linear transformation $T : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ by $T(X) = AX -XA$. Find the matrix of $T$ in the basis $B = \{(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}), (\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\}$.

Solution. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We compute

$T(E_1) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}) (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) - (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) (\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}) = -bE_2 + cE_3$.

Similarly,

$T(E_2) = -cE_1 + (a - d)E_2 + cE_4$
$T(E_3) = bE_1 + (d - a)E_3 - bE_4$
$T(E_4) = bE_2 - cE_3$.

Thus we obtain

$[T]_B = \begin{pmatrix} 0 & -c & b & 0 \\ -b & a - d & 0 & b \\ c & 0 & d - a & -c \\ 0 & c & -b & 0 \end{pmatrix}$.

2. Let $\mathcal{M}_n$ be the vector space of complex $(n \times n)$-matrices and consider the linear operator $T : V \rightarrow V$ defined by $T(A) = A^\top$.

(a) Show that $\pm 1$ are the only eigenvalues of $T$.

(b) Describe the eigenvectors corresponding to each eigenvalue of $T$.

(c) Find a basis $B$ for $\mathcal{M}_2$ such that $[T]_B$ is a diagonal matrix.

Solution.

(a) To find the eigenvalues of $T$, we must find $\lambda \in \mathbb{C}$ such that $T(A) = A^\top = \lambda A$. For nonzero $A = [a_{j,k}] \in \mathcal{M}_n$, we obtain the equations $a_{k,j} = \lambda a_{j,k}$ for $1 \leq j \leq n$ and $1 \leq k \leq n$. Hence, we have $a_{k,j} = \lambda a_{j,k} = \lambda(\lambda a_{k,j})$ and $(\lambda^2 - 1)a_{k,j} = 0$. Since $A \neq 0$, we deduce that $\lambda^2 - 1 = 0$ and $\lambda = \pm 1$ as required.
Another way to see that \( \lambda = \pm 1 \) is to compute

\[
A = (A^T)^T = (\lambda A)^T = \lambda A^T = \lambda (\lambda A) = \lambda^2 A,
\]

i.e. we get \((\lambda^2 - 1)A = 0\) which together with \(A \neq 0\) implies that \(\lambda^2 = 1\).

(b) The eigenvectors corresponding to the eigenvalue \( \lambda = 1 \) satisfy the equation \( A^T = A \); in other words, these eigenvectors are the nonzero symmetric matrices. The eigenvectors corresponding to \( \lambda = -1 \) satisfy \( A^T = -A \); they are the nonzero skew-symmetric matrices.

(c) Let \( E_{j,k} \) denote the matrix units. Since

\[
T(E_{1,1}) = (E_{1,1})^T = E_{1,1} \quad T(E_{1,2} + E_{2,1}) = (E_{1,2} + E_{2,1})^T = E_{1,2} + E_{2,1} \\
T(E_{j,j}) = (E_{2,2})^T = E_{2,2} \quad T(E_{1,2} - E_{2,1}) = (E_{1,2} - E_{2,1})^T = (-1)(E_{1,2} - E_{2,1}),
\]

we see that \( E_{1,1}, E_{2,2}, E_{1,2} + E_{2,1} \) are eigenvectors corresponding to the eigenvalue 1 and \( E_{1,2} - E_{2,1} \) is an eigenvectors corresponding to the eigenvalue \(-1\). The equation

\[
0 = aE_{1,1} + bE_{2,2} + c(E_{1,2} + E_{2,1}) + d(E_{1,2} - E_{2,1})
\]

implies that \( a = 0 \), \( b = 0 \) and \( c + d = 0 = c - d \), so \( a = b = c = d = 0 \). It follows that \( \mathcal{B} := (E_{1,1}, E_{2,2}, E_{1,2} + E_{2,1}, E_{1,2} - E_{2,1}) \) is linearly independent. Since \( \mathcal{B} \) has 4 elements and \( \text{dim} \mathcal{M}_n = 4 \), \( \mathcal{B} \) is a basis consisting of eigenvectors of \( T \) and

\[
[T]_{\mathcal{B}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.
\]

3. Let \( \mathcal{T}_1 \) be the \( \mathbb{C} \)-vector space with basis \( \mathcal{B} = (1, \cos(x), \sin(x)) \). Define \( J : \mathcal{T}_1 \to \mathcal{T}_1 \) by \( (Jf)(x) = \int_0^\pi f(x-t) \, dt \). Show that \( J \) is diagonalizable and find an eigenbasis.
Solution. Since
\[
(J1)(x) = \int_0^\pi 1 \, dt = \pi
\]
\[
(J \cos)(x) = \int_0^\pi \cos(x-t) \, dt = [-\sin(x-t)]_0^\pi = -\sin(x-\pi) + \sin(x) = 2 \sin(x)
\]
\[
(J \sin)(x) = \int_0^\pi \sin(x-t) \, dt = [\cos(x-t)]_0^\pi = \cos(x-\pi) - \cos(x) = -2 \cos(x),
\]
we have \(A = [J]_{\mathbb{H}} = \begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} \). To find the eigenvalues of \(J\) we recall that \(\lambda\) is an eigenvalue of \(J\) if and only if the matrix \([J]_{\mathbb{H}} - \lambda I\) is not invertible. Notice first that, if \(\lambda = \pi\), it is not invertible, because its first row is zero. Assuming now that \(\lambda \neq \pi\), we use row reduction to determine when \([J]_{\mathbb{H}} - \lambda I\) is not invertible:

\[
\begin{bmatrix}
\pi - \lambda & 0 & 0 \\
0 & -\lambda & -2 \\
0 & 2 & -\lambda \\
\end{bmatrix}
\begin{array}{c}
\frac{1}{\pi - \lambda} R1 \\
R2 \leftrightarrow R3 \\
R3 + \frac{1}{2}R2
\end{array}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & -\lambda \\
0 & -\lambda & -2 \\
\end{bmatrix}
\begin{array}{c}
R3 + \frac{1}{2}R2 \\
R2 \\
R1
\end{array}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & -\lambda \\
0 & 0 & -2 - \frac{\lambda^2}{2}
\end{bmatrix}.
\]

The last matrix is invertible if and only if \(-2 - \frac{\lambda^2}{2} \neq 0\), i.e., if and only if \(\lambda \neq \pm 2i\). So the eigenvalues of \(J\) are \(-2i, 2i\) and \(\pi\). To find an eigenvector \(u_1\) with and eigenvalue \(-2i\) we calculate

\[
A + 2i I = \begin{bmatrix}
\pi + 2i & 0 & 0 \\
0 & 2i & -2 \\
0 & 2 & 2i
\end{bmatrix}
\]

and find that we can take \(u_1 = [0, 1, i]_{\mathbb{H}} = \cos(x) + i \sin(x)\). Similarly, \(u_2 = \cos(x) - i \sin(x)\) is an eigenvector with eigenvalue \(2i\) and \(u_3 = 1\) is an eigenvector with eigenvalue \(\pi\). Therefore, \(C = (\cos(x) + i \sin(x), \cos(x) - i \sin(x), 1)\) is a basis of eigenvectors for \(J\). \(\square\)