Solutions #20

1. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Define the linear transformation \( T : \mathcal{M}_2 \to \mathcal{M}_2 \) by \( T(X) = AX + XA \). Find the matrix of \( T \) in the basis

\[ \mathcal{B} = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \).

Solution. Let \( E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). We compute

\[ T(E_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2a & b \\ c & 0 \end{bmatrix} = 2aE_1 + bE_2 + cE_3. \]

Similarly,

\[ T(E_2) = cE_1 + (a + d)E_2 + cE_4 \]
\[ T(E_3) = bE_1 + (a + d)E_3 + bE_4 \]
\[ T(E_4) = bE_2 + cE_3 + 2dE_4. \]

Thus we obtain

\[ [T]_{\mathcal{B}} = \begin{bmatrix} 2a & c & b & 0 \\ b & a + d & 0 & b \\ c & 0 & a + d & c \\ 0 & c & b & 2d \end{bmatrix}. \] □

2. Let \( \mathcal{M}_n \) be the vector space of complex \((n \times n)\)-matrices and consider the linear operator \( T : \mathcal{M}_n \to \mathcal{M}_n \) defined by \( T(A) = -A^T \).

(a) Show that \( \pm 1 \) are the only eigenvalues of \( T \).

(b) Describe the eigenvectors corresponding to each eigenvalue of \( T \).

(c) Find a basis \( \mathcal{B} \) for \( \mathcal{M}_2 \) such that \( [T]_{\mathcal{B}} \) is a diagonal matrix.

Solution.

(a) To find the eigenvalues of \( T \), we must find \( \lambda \in \mathbb{C} \) such that \( T(A) = -A^T = \lambda A \). For nonzero \( A = [a_{j,k}] \in \mathcal{M}_n \), we obtain the equations \( a_{k,j} = -\lambda a_{j,k} \) for \( 1 \leq j \leq n \) and \( 1 \leq k \leq n \). Hence, we have \( a_{k,j} = -\lambda a_{j,k} = -\lambda (-\lambda a_{k,j}) = \lambda^2 a_{k,j} \). Thus \( (\lambda^2 - 1)a_{k,j} = 0 \). Since \( A \neq 0 \), not all \( a_{k,j} \) can equal zero, and we deduce that \( \lambda^2 - 1 = 0 \), i.e., \( \lambda = \pm 1 \) as required.
Another way to see that $\lambda = \pm 1$ is to compute
\[ A = (A^T)^T = (-\lambda A)^T = -\lambda A^T = -\lambda (-\lambda A) = \lambda^2 A, \]
i.e. we get $(\lambda^2 - 1)A = 0$ which together with $A \neq 0$ implies that $\lambda^2 = 1$.

(b) The eigenvectors corresponding to the eigenvalue $\lambda = 1$ satisfy the equation $A^T = -A$; in other words, these eigenvectors are the nonzero skew-symmetric matrices. The eigenvectors corresponding to $\lambda = 1$ satisfy $-A^T = -A$; they are the nonzero symmetric matrices.

(c) According to (b), a basis of eigenvectors should consists of skew-symmetric and symmetric matrices. Consider the set $B := \{ E_{1,2} - E_{2,1}, E_{1,1}, E_{2,2}, E_{1,2} + E_{2,1} \}$. It is a linearly independent set (Why?) consisting of 4 vectors in the four-dimensional space $\mathbb{M}_2$; hence $B$ is a basis of $\mathbb{M}_2$. Moreover,
\[ [T]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \]

3. Let $\mathcal{T}$ be the $\mathbb{C}$-vector space with basis $B = \{ 1, \cos x, \sin x \}$. Define $J: \mathcal{T} \to \mathcal{T}$ by $(Jf)(x) = \int_0^\pi f(x+t) \, dt$. Show that $J$ is diagonalizable and find a basis of $\mathcal{T}$ consisting of eigenvectors of $J$.

Solution. Since
\[ (J1)(x) = \int_0^\pi 1 \, dt = \pi \]
\[ (J\cos)(x) = \int_0^\pi \cos(x+t) \, dt = [\sin(x+t)]_0^\pi = \sin(x+\pi) - \sin(x) = -2\sin(x) \]
\[ (J\sin)(x) = \int_0^\pi \sin(x+t) \, dt = [-\cos(x+t)]_0^\pi = -\cos(x+\pi) + \cos(x) = 2\cos(x), \]
we have $A = [J]_B = \begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$.

Similarly to the example from class, we notice that
\[ A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2i \\ -2 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ -2i \\ -2i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]
Since the vectors \[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}
\] are linearly independent (Why?) eigenvectors of \(A = [J]_{B}\), we conclude that \(A\) is diagonalizable with eigenvalues \(\pi, 2i,\) and \(-2i\). Going back to \(J\), it is diagonalizable with eigenvalues \(\pi, 2i,\) and \(-2i\) and corresponding basis of eigenvectors \(\mathcal{C} = \{1, \cos x + i \sin x, \cos x - i \sin x\}\). \qed