Solutions #21

1. For a matrix $A$ denote the characteristic polynomial of $A$ by $\xi_A(t)$, i.e.

$$\xi_A(t) = \det(A - tI),$$

where $I$ is the identity matrix.

(a) Prove that $\xi_A(t^2) = \xi_A(t)\xi_A(-t)$.

(b) Given that $\xi_A(t) = t^4 + t + 1$, find $\xi_A(t^2)$.

Solution.

(a) Calculate

$$\xi_A(t^2) = \det(a^2 - t^2I) = \det((A - tI)(A + tI)) = \det(A - tI)\det(A + tI) = \xi_A(t)\xi_A(-t).$$

(b) Using (a) we calculate

$$\xi_A(t^2) = \xi_A(t)\xi_A(-t) = (t^4 + t + 1)(t^4 - t + 1) = t^8 + 2t^4 - t^2 + 1 = (t^2)^4 + 2(t^2)^2 - (t^2) + 1,$$

which implies that $\xi_A(t^2) = t^4 + 2t^2 - t + 1$. $\square$

2. Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ be three vectors in $\mathbb{R}^3$. Recall that the oriented volume $V(u, v, w)$ of the parallelepiped spanned by $u, v, w$ equals $(u \times v) \cdot w$. (See p. 45 for the definition of the cross product.)

(a) Assuming the formula $V(u, v, w) = (u \times v) \cdot w$, prove that

$$V(u, v, w) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

(b) If $A$ is a $3 \times 3$–matrix, prove that

$$V(Au, Av, Aw) = \det(A)V(u, v, w).$$

Solution.
(a) Using that $u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$ we calculate

$$(u \times v) \cdot w = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \cdot (w_1, w_2, w_3)$$

$$= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3$$

$$= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3 = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$ 

(b) Using that $\det A^T = \det A$ we can write the formula from (a) by

$$V(u, v, w) = \det [u, v, w],$$

where $[u, v, w]$ denotes the matrix whose columns are the vectors $u, v, w$ respectively. Noticing that

$$[Au, Av, Aw] = A[u, v, w]$$

we calculate

$$V(Au, Av, Aw) = \det [Au, Av, Aw] = \det(A[u, v, w])$$

$$= \det(A) \det([u, v, w]) = \det(A)V(u, v, w).$$  

\hfill \Box

3. Let $T : U \to U$ be a linear transformation and let $B$ be a basis of $U$. Define the determinant $\det(T)$ of $T$ as

$$\det(T) = \det([T]_B).$$

Show that $\det(T)$ is well–defined, i.e. that it does not depend on the choice of the basis $B$. Prove that $T$ is invertible if and only if $\det(T) \neq 0$. If $T$ is invertible, show that

$$\det(T^{-1}) = \frac{1}{\det(T)}.$$  

Solution. Let $C$ be another basis of $U$ and let $P = P_{B \leftarrow C}$. Then $[T]_C = P^{-1}[T]_B P$ which implies

$$\det([T]_C) = \det(P^{-1}[T]_B P) = \det(P^{-1}) \det([T]_B) \det(P) = \frac{1}{\det(P)} \det([T]_B) \det(P) = \det([T]_B).$$
For the second part we see that $T$ is invertible if and only if $[T]_\mathcal{B}$ is invertible if and only if $\det([T]_\mathcal{B}) \neq 0$ if and only if $\det(T) \neq 0$. Finally,

$$\det(T^{-1}) = \det([T^{-1}]_\mathcal{B}) = \det(([T]_\mathcal{B})^{-1}) = \frac{1}{\det([T]_\mathcal{B})} = \frac{1}{\det(T)}.$$

$\square$