Solutions #24

1(a). Prove that

$$\langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

defines an inner product in the space $\mathbb{R}[x]_{\leq 2}$ of polynomials of degree at most 2. Is $\langle p(x), q(x) \rangle$ an inner product in the space $\mathbb{R}[x]_{\leq 3}$?

**Solution.** We check the properties of inner product for the space $\mathbb{R}[x]_{\leq 2}$.

$$\langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

$$= q(0)p(0) + q(1)p(1) + q(2)p(2) = \langle q(x), p(x) \rangle.$$ 

$$\langle p(x) + p_2(x), q(x) \rangle = (p_1(0) + p_2(0))q(0) + (p_1(1) + p_2(1))q(1) + (p_1(2) + p_2(2))q(2)$$

$$= (p_1(0)q(0) + p_1(1)q(1) + p_1(2)q(2)) + (p_2(0)q(0) + p_2(1)q(1) + p_2(2)q(2))$$

$$= \langle p_1(x), q(x) \rangle + \langle p_2(x), q(x) \rangle.$$ 

$$\langle \lambda p(x), q(x) \rangle = (\lambda p(0))q(0) + (\lambda p(1))q(1) + (\lambda p(2))q(2)$$

$$= \lambda (p(0)q(0) + p(1)q(1) + p(2)q(2)) = \langle \lambda p(x), q(x) \rangle.$$ 

$$\langle p(x), p(x) \rangle = p(0)^2 + p(1)^2 + p(2)^2 \geq 0.$$ 

Finally, we need to check that $\langle p(x), p(x) \rangle = 0$ implies $p(x) = 0$. Notice that $\langle p(x), p(x) \rangle = 0$ implies that $p(0) = p(1) = p(2) = 0$. But a polynomial of degree $\leq 2$ can have at most two roots, we conclude that $p(x) = 0$.

To see that $\langle p(x), q(x) \rangle$ is not an inner product in $\mathbb{R}[x]_{\leq 3}$ we notice that if we set $p(x) = x(x-1)(x-2)$, then $\langle p(x), p(x) \rangle = 0$ while $p(x) \neq 0$. 

\[\square\]

1(b). Prove that

$$\langle A, B \rangle = -\text{tr}(AB)$$

defines an inner product on the space of skew–symmetric $n \times n$–matrices.

**Solution.** We check the properties of inner product.

$$\langle A, B \rangle = -\text{tr}(AB) = -\text{tr}(BA) = \langle A, B \rangle.$$ 

$$\langle A + A_2, B \rangle = -\text{tr}(A_1 + A_2)B = -\text{tr}(A_1B + A_2B) = -\text{tr}(A_1B) - \text{tr}(A_2B) = \langle A, B \rangle + \langle A_2, B \rangle.$$ 

$$\langle \lambda A, B \rangle = -\text{tr}(\lambda A)B = -\text{tr}(\lambda AB) = -\lambda \text{tr}(AB) = \lambda \langle A, B \rangle.$$ 

Finally, using the formula

$$\text{tr}(AB) = \sum_{ij} a_{ij}b_{ji},$$

where $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $n \times n$–matrices, we obtain

$$\langle A, A \rangle = -\text{tr}(AA) = -\sum_{i,j} a_{ij}a_{ji} = -\sum_{i,j} a_{ij}(-a_{ij}) = \sum_{i\neq j} a_{ij}^2 \geq 0$$

and $\langle A, A \rangle = \sum_{i\neq j} a_{ij} = 0$ implies $a_{ij} = 0$ for every $i \neq j$, i.e. $A = 0$. 

\[\square\]
2. Let \( \langle \vec{u}, \vec{v} \rangle \) be an inner product in \( \mathbb{R}^n \).

(a) Prove that there exists a symmetric \( n \times n \)–matrix \( A \) such that
\[
\langle \vec{u}, \vec{v} \rangle = \vec{u}^T A \vec{v}.
\]
(b) If \( n = 2 \), prove that \( a_{11} > 0 \) and \( \det(A) > 0 \), where \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) is the matrix from (a).

**Solution.**

(a) Let \( \mathcal{B} = \{e_1, \ldots, e_n\} \) be the standard basis of \( \mathbb{R}^n \) and let \( u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \) and \( v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \), i.e. \( u = u_1 e_1 + \ldots + u_n e_n \) and \( v = v_1 e_1 + \ldots + v_n e_n \). Using the linearity of dot product we obtain
\[
\langle u, v \rangle = \langle u_1 e_1 + \ldots + u_n e_n, v_1 e_1 + \ldots + v_n e_n \rangle = \sum_{ij} u_i v_j \langle e_i, e_j \rangle.
\]
Set \( a_{ij} = \langle e_i, e_j \rangle \). Then \( a_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = a_{ji} \) which proves that \( A = [a_{ij}] \) is a symmetric matrix. Finally,
\[
\langle u, v \rangle = \sum_{ij} u_i v_j \langle e_i, e_j \rangle = \sum_{ij} u_i v_j a_{ij} = \sum_{ij} u_i a_{ij} v_j = u^T A v.
\]

(b) We first notice that \( a_{11} = \langle e_1, e_1 \rangle > 0 \). Next we apply Cauchy–Schwarz inequality to the vectors \( e_1 \) and \( e_2 \) to get
\[
\langle e_1, e_2 \rangle^2 \leq \| e_1 \|^2 \| e_2 \|^2,
\]
which after substitution becomes
\[
a_{12}^2 \leq a_{11} a_{22},
\]
i.e. \( \det(A) = a_{11} a_{22} - a_{12} a_{21} \geq 0 \). To complete the proof that \( \det(A) > 0 \) we need to make sure that \( \det(A) \neq 0 \). Assume that \( \det(A) = 0 \). Then there exists \( u \neq 0 \) such that \( Au = 0 \), which would imply \( \langle u, u \rangle = u^T A u = u^T (Au) = 0 \), which contradicts the assumption that \( u \neq 0 \). \( \square \)

3. Let \( \langle \vec{u}, \vec{v} \rangle \) be an inner product in a real vector space \( U \). Given \( k \) vectors \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \) in \( U \), set \( a_{ij} := \langle \vec{u}_i, \vec{u}_j \rangle \).

(a) Prove that the matrix \( A = (a_{ij}) \) is invertible if and only if \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \) are linearly independent.

(b) Prove that, for any scalars \( x_1, x_2, \ldots, x_k \in \mathbb{R} \), we have
\[
\sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} x_i x_j \geq 0.
\]

**Solution.**
(a) First we prove that, if \( A \) is invertible, then \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k \) are linearly independent. Assume that \( A \) is invertible and let

\[
\begin{align*}
\sum_{i=1}^k c_i \tilde{u}_i &= 0.
\end{align*}
\]

We will show that \( c_1 = c_2 = \ldots = c_k = 0 \). For every \( 1 \leq j \leq k \) we have

\[
0 = \langle \bar{0}, \tilde{u}_j \rangle = \langle \bar{0}, c_1 \tilde{u}_1 + c_2 \tilde{u}_2 + \ldots + c_k \tilde{u}_k \rangle
= c_1 \langle \bar{0}, \tilde{u}_1 \rangle + c_2 \langle \bar{0}, \tilde{u}_2 \rangle + \ldots + c_k \langle \bar{0}, \tilde{u}_k \rangle
= c_1 a_j c_1 + c_2 a_j c_2 + \ldots + c_k a_j c_k.
\]

Combining the above equations (for \( j = 1, 2, \ldots, k \)), we see that \( A\tilde{c} = \bar{0} \), where \( \tilde{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \).

Since \( A \) is an invertible matrix, we conclude that \( \tilde{c} = \bar{0} \), which completes the proof that \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k \) are linearly independent.

Now we prove that, if \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k \) are linearly independent, then \( A \) is invertible. Let \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k \) be linearly independent and, for the sake of contradiction, assume that \( A \) is not invertible. Let \( \tilde{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \) be a nonzero solution of \( A\tilde{c} = \bar{0} \). (Why does \( \tilde{c} \) exist?) This means that, for every \( 1 \leq j \leq k \), we have

\[
0 = a_j c_1 + a_j c_2 + \ldots + a_j c_k = c_1 \langle \bar{0}, \tilde{u}_j \rangle + c_2 \langle \bar{0}, \tilde{u}_j \rangle + \ldots + c_k \langle \bar{0}, \tilde{u}_j \rangle
= \langle \tilde{u}_j, c_1 \tilde{u}_1 + c_2 \tilde{u}_2 + \ldots + c_k \tilde{u}_k \rangle.
\]

Setting \( \bar{u} := c_1 \tilde{u}_1 + c_2 \tilde{u}_2 + \ldots + c_k \tilde{u}_k \), we see that the above means that \( \langle \bar{u}, \tilde{u}_j \rangle = 0 \) for every \( j = 1, 2, \ldots, k \). Then

\[
\langle \bar{u}, \bar{u} \rangle = \langle c_1 \tilde{u}_1 + c_2 \tilde{u}_2 + \ldots + c_k \tilde{u}_k, \bar{u} \rangle = c_1 \langle \bar{u}, \tilde{u}_1 \rangle + c_2 \langle \bar{u}, \tilde{u}_2 \rangle + \ldots + c_k \langle \bar{u}, \tilde{u}_k \rangle = 0.
\]

This equation implies that \( \bar{u} = \bar{0} \), which is a contradiction with the assumption that \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k \) are linearly independent because \( \tilde{c} \neq \bar{0} \).

(b) Set \( \bar{u} := x_1 \tilde{u}_1 + x_2 \tilde{u}_2 + \ldots + x_k \tilde{u}_k \). Using that the inner product is bilinear we show as above that

\[
\sum_{i=1}^k \sum_{j=1}^k a_{ij} x_i x_j = \langle \bar{u}, \bar{u} \rangle \geq 0
\]
\( \square \).