Solutions Test III, Version 1

Part I. For each of the following sets answer whether it is a vector space or not. In all cases the operations are the natural ones. Give a very short argument: you do not need to verify all properties. Examples of good arguments are: "No, because it is not closed under ..." or "Yes, because it is a subset of ... closed under ... and under ...".

(a) Matrices of size $3 \times 3$ of rank 3.
(b) Continuous functions $f(x)$ such that $\lim_{x \to \infty} f(x) = 0$.
(c) $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$.
(d) $\{(x,y) \in \mathbb{R}^2 \mid x^2 = y^2\}$.
(e) The set of skew-symmetric $n \times n$-matrices. A matrix is skew-symmetric if $A^T = -A$.

Solution.

(a) No. Reasons: It does not contain the zero matrix. Also, the sum of two matrices of rank 3 may be a matrix of lower rank.

(b) Yes. If $f(x)$ and $g(x)$ are both continuous with $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to \infty} g(x) = 0$ and $\lambda \in \mathbb{R}$, then both $f(x) + g(x)$ and $\lambda f(x)$ are continuous and $\lim_{x \to \infty} f(x) + g(x) = 0$ and $\lim_{x \to \infty} \lambda f(x) = 0$. Hence this set is a subspace of the vector space of all functions (from $\mathbb{R}$ to $\mathbb{R}$). It is also a subspace of the space of all continuous functions.

(c) No. Reason: It does not contain the opposite vectors: the vector $(1,1)$ is in the set, while $(-1,-1)$ is not. More generally, multiplying by a negative scalar gets us outside of the set: $-3(1,2) = (-3,-6)$.

(d) No. Reason: It is not closed under addition: $(1,1)$ and $(1,-1)$ are in the set while $(1,1) + (1,-1) = (2,0)$ is not.

(e) Yes. If $A$ and $B$ are skew-symmetric matrices and $\lambda$ is a scalar, then both $A + B$ and $\lambda A$ are skew-symmetric matrices. Hence this set is a subspace of the vector space of $n \times n$-matrices.

Part II(A). Let $C(\mathbb{R})$ denote the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Is the set $W$ of even continuous functions a subspace of $C(\mathbb{R})$? Justify your answer! Recall that $f$ is even if $f(-x) = f(x)$ for every $x \in \mathbb{R}$.

Solution. Yes, it is. Let $f,g \in W$, i.e., they are even continuous functions, and let $\lambda \in \mathbb{R}$. Both $f + g$ and $\lambda f$ are continuous functions and

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$$

and

$$(\lambda f)(-x) = \lambda f(-x) = \lambda f(x) = (\lambda f)(x).$$

This completes the verification that $f + g \in W$ and $\lambda f \in W$, proving that $W$ is a subspace of $C(\mathbb{R})$.

Part II(B). Show that the functions $f_1(x) = x, f_2(x) = \sin x$, and $f_3(x) = \sqrt{1 + x^2}$ are linearly independent in $C(\mathbb{R})$.

Solution. Assume that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.$$
This means that, for every \( x \), we have
\[
c_1x + c_2 \sin x + c_3 \sqrt{1+x^2} = 0.
\]
Setting \( x = 0 \), \( x = \pi \) and \( x = 1 \), we obtain respectively
\[
\begin{align*}
c_1 \times 0 + c_2 \times 0 + c_3 \times 1 &= 0 \\
c_1 \times \pi + c_2 \times 0 + c_3 \times \sqrt{1+\pi^2} &= 0 \\
c_1 \times 1 + c_2 \times \sin 1 + c_3 \times \sqrt{2} &= 0.
\end{align*}
\]
This is the following linear system for \( c_1, c_2, c_3 \):
\[
\begin{align*}
\pi c_1 &= 0 \\
c_2 + \sqrt{\pi^2+1} c_3 &= 0 \\
c_1 + \sin(1) c_2 + \sqrt{2} c_3 &= 0.
\end{align*}
\]
Since the only solution of this system is \( c_1 = c_2 = c_3 = 0 \), we conclude that \( f_1, f_2, f_3 \) are linearly independent. \( \square \)

**Part III.** Consider the polynomials
\[
p_1(x) = 1+x, \quad p_2(x) = x+x^2, \quad p_3(x) = 1-x^2, \quad p_4(x) = (1+x)^2, \quad p_5(x) = 1+x^2.
\]
Find a subset of the set \( \{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\} \) which is a basis of
\[
\text{span}\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}.
\]
**Solution.** Since \( p_1 \neq 0 \), the set \( \{p_1\} \) is linearly independent and thus we keep \( p_1 \).
Next, \( p_1 \) and \( p_2 \) are linearly independent (Why?), thus we keep \( p_2 \) as well.
Since \( p_3 = p_1 - p_2 \), the set \( \{p_1, p_2, p_3\} \) is linearly dependent, so we throw \( p_3 \) away.
Since \( p_4 = p_1 + p_2 \), the set \( \{p_1, p_2, p_4\} \) is linearly dependent, so we throw \( p_4 \) away as well.
Since \( \{p_1, p_2, p_5\} \) is a linearly independent set (Why?), we keep \( p_5 \).
**Answer.** \( \{p_1, p_2, p_5\} \) is a basis of \( \text{span}\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\} \).

**Remark.** There are other correct answers but this is the one obtained by following the algorithm from class. If you gave a different answer, you needed to provide a justification for it. \( \square \)

**Part IV.**
(a) Show that \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) given by \( T(p(x)) = p(x^2) \) is a linear transformation.
(b) Calculate \( (S \circ T)(x-1) \) where \( T \) is the linear transformation from (a) above and \( S : \mathbb{R}[x] \to \text{Mat}_{2 \times 2}(\mathbb{R}) \) is given by
\[
S(p(x)) = \begin{bmatrix} p(2) & \int_0^1 p(x)dx \\ 0 & p(2) + p(-2) \end{bmatrix}.
\]
**Solution.**
(a) Let \( p, q \in \mathbb{R}[x] \) and \( \lambda \in \mathbb{R} \). For any \( x \in \mathbb{R} \) we have
\[
T(p+q)(x) = (p+q)(x^2) = p(x^2) + q(x^2) = T(p)(x) + T(q)(x) = (T(p) + T(q))(x)
\]
and
\[
T(\lambda p)(x) = (\lambda p)(x^2) = \lambda p(x^2) = \lambda T(p)(x) = (\lambda T(p))(x).
\]
Since the identities above hold for every real \( x \), we conclude that
\[
T(p + q) = T(p) + T(q) \quad \text{and} \quad T(\lambda p) = \lambda T(p),
\]
which proves that \( T \) is a linear transformation.

(b) Since \( T(x - 1) = x^2 - 1 \), we obtain
\[
(S \circ T)(x - 1) = S(T(x - 1)) = S(x^2 - 1) = \begin{bmatrix}
2^2 - 1 \\
0
\end{bmatrix} + \int_0^1 x^2 - 1 \, dx \\
(2^2 - 1) + ((-2)^2 - 1) = \begin{bmatrix}
3 \\
0
\end{bmatrix} - \frac{2}{3}.
\]
\[
\square
\]

Part V. Let \( T : \mathbb{F}^3 \to \mathbb{F}^4 \) be a linear transformation with image
\[
\text{Im} T = \{ r \begin{bmatrix}
1 \\
-2 \\
0 \\
4
\end{bmatrix} + s \begin{bmatrix}
0 \\
3 \\
-2 \\
2
\end{bmatrix} | r, s \in \mathbb{F}\}.
\]

(a) Write down a basis of \( \text{Im} T \).

(b) Can \( \text{Ker}(T) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \)? Explain!

Solution. (a) The vectors \( \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 3 \\ -2 \\ 2 \end{bmatrix} \) span \( \text{Im} T \) and they are linearly independent (Why?), so they form a basis of \( \text{Im} T \).

(b) Since \( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \) are linearly independent, they form a basis of \( \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \).

In particular, the dimension of \( \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \) is 2. By (a) we have that \( \dim \text{Im} T = 2 \) and, by the dimension theorem,
\[
\dim \text{Im} T + \dim \text{Ker} T = \dim \mathbb{F}^3.
\]
The last implies that \( \dim \text{Ker} T = 1 \), hence it cannot equal \( \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \) which is two dimensional. \( \square \)
Part I. For each of the following sets answer whether it is a vector space or not. In all cases the operations are the natural ones. Give a very short argument: you do not need to verify all properties. Examples of good arguments are: "No, because it is not closed under ..." or "Yes, because it is a subset of ... closed under ... and under ...".

(a) Matrices of size $3 \times 3$ whose rank is no greater than 2.
(b) Continuous functions $f(x)$ such that $\lim_{x \to \infty} f(x) = 1$.
(c) $\{(x, y) \in \mathbb{R}^2 | xy \geq 0\}$.
(d) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 0\}$.
(e) The set of invertible $n \times n$-matrices.

Solution.

(a) No. Reasons: It is not closed under addition: both
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
have rank 2, while their sum
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
has rank 3.

(b) No. Reasons: The zero function is not contained in this set. Also this set is not closed neither under addition nor under multiplication by a scalar.

(c) No. Reason: It is not closed under addition: $(1, 0)$ and $(0, -1)$ are in the set while $(1, 0) + (0, -1) = (1, -1)$ is not.

(d) Yes. Reason: This set consist of the point $(0, 0)$ only which is a subspace of $\mathbb{R}^2$, namely the zero subspace.

(e) No. Reasons: The zero matrix is not invertible. Neither is the set of invertible matrices closed under addition (give an example!). □

Part II(A). Let $C(\mathbb{R})$ denote the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Is the set $W$ of $2\pi$-periodic continuous functions a subspace of $C(\mathbb{R})$?

Solution. Yes, it is. Let $f, g \in W$, i.e., they are $2\pi$-periodic continuous functions, and let $\lambda \in \mathbb{R}$. Both $f + g$ and $\lambda f$ are continuous functions and
\[
(f + g)(x + 2\pi) = f(x + 2\pi) + g(x + 2\pi) = f(x) + g(x) = (f + g)(x)
\]
and
\[
(\lambda f)(x + 2\pi) = \lambda f(x + 2\pi) = \lambda f(x) = (\lambda f)(x).
\]
This completes the verification that $f + g \in W$ and $\lambda f \in W$, proving that $W$ is a subspace of $C(\mathbb{R})$. □

Part II(B). Show that the functions $f_1(x) = x, f_2(x) = \cos x$, and $f_3(x) = \frac{x^2}{1 + x^2}$ are linearly independent in $C(\mathbb{R})$.

Solution. Assume that
\[
c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.
\]
This means that, for every $x$, we have
\[ c_1 x + c_2 \cos x + c_3 \frac{x^2}{1+x^2} = 0. \]
Setting $x = 0$, we obtain that $c_2 = 0$ and hence
\[ c_1 x + c_3 \frac{x^2}{1+x^2} = x(c_1 + c_3 \frac{x}{1+x^2}) = 0. \]
This means that, for $x \neq 0$, we have $c_1 + c_3 \frac{x}{1+x^2} = 0$. Since $c_1 + c_3 \frac{x}{1+x^2}$ is continuous, we conclude that $c_1 + c_3 \frac{x}{1+x^2} = 0$ for every $x$. Setting $x = 0$ in the last equation, we obtain $c_1 = 0$ and $c_3 \frac{x}{1+x^2} = 0$, which for $x = 1$ implies $c_3 = 0$.

To summarize, the assumption that $$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$ led us to the conclusion that $c_1 = c_2 = c_3 = 0$, proving that $f_1, f_2, f_3$ are linearly independent. □

**Part III.** Consider the polynomials
\[ p_1(x) = 1 + x, \quad p_2(x) = 1 + x^2, \quad p_3(x) = 3 + 2x + x^2, \quad p_4(x) = x + x^2, \quad p_5(x) = 1 + x + x^2. \]
Find a subset of the set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$ which is a basis of $\text{span}\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$.

**Solution.** Since $p_1 \neq 0$, the set $\{p_1\}$ is linearly independent and thus we keep $p_1$.
Next, $p_1$ and $p_2$ are linearly independent (Why?), thus we keep $p_2$ as well.
Since $p_3 = 2p_1 + p_2$, the set $\{p_1, p_2, p_3\}$ is linearly dependent, so we throw $p_3$ away.
Since $\{p_1, p_2, p_4\}$ is a linearly independent set (Why?), we keep $p_4$.
Since $p_5 = \frac{1}{2}p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_4$, the set $\{p_1, p_2, p_4, p_5\}$ is linearly dependent, so we throw $p_5$ away.
**Answer.** $\{p_1, p_2, p_4\}$ is a basis of $\text{span}\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$.

**Remark.** There are other correct answers but this is the one obtained by following the algorithm from class. If you gave a different answer, you needed to provide a justification for it. □

**Part IV.**

(a) Show that $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $T(p(x)) = p(x-2)$ is a linear transformation.
(b) Calculate $(S \circ T)(x^2)$ where $T$ is the linear transformation from (a) above and $S : \mathbb{R}[x] \to \text{Mat}_{2 \times 2}(\mathbb{R})$ is given by
\[ S(p(x)) = \begin{bmatrix} p(1) & \int_0^1 p(x) \, dx \\ 0 & p(1) - p(-1) \end{bmatrix}. \]

**Solution.**

(a) Let $p, q \in \mathbb{R}[x]$ and $\lambda \in \mathbb{R}$. For any $x \in \mathbb{R}$ we have
\[ T(p + q)(x) = (p + q)(x-2) = p(x-2) + q(x-2) = T(p)(x) + T(q)(x) = (T(p) + T(q))(x) \]
and
\[ T(\lambda p)(x) = (\lambda p)(x-2) = \lambda p(x-2) = \lambda T(p)(x) = (\lambda T(p))(x). \]
Since the identities above hold for every real \( x \), we conclude that 

\[
T(p + q) = T(p) + T(q) \quad \text{and} \quad T(\lambda p) = \lambda T(p),
\]

which proves that \( T \) is a linear transformation.

(b) Since \( T(x^2) = (x - 2)^2 \), we obtain

\[
(S \circ T)(x^2) = S(T(x)) = S((x - 2)^2) = \left[ \frac{(1 - 2)^2}{(1 - 2)^2 - (-1 - 2)^2} \right] = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right]. \]

\[ \square \]

Part V. Let \( T : \mathbb{F}^4 \to \mathbb{F}^4 \) be a linear transformation with kernel

\[
\ker(T) = \{r \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} + s \begin{bmatrix} 0 \\ 3 \\ -2 \\ 2 \end{bmatrix} \mid r, s \in \mathbb{F} \}.
\]

(a) Write down a basis of \( \ker(T) \).

\[
\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}
\]

(b) Can \( \text{Im}(T) = \text{span}\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \}? Explain!

Solution. (a) The vectors \( \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 3 \\ -2 \\ 2 \end{bmatrix} \) span \( \ker(T) \) and they are linearly independent (Why?), so they form a basis of \( \ker(T) \).

(b) Since \( \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \) are linearly independent, they form a basis of \( \text{span}\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \} \) is 3. By (a) we have that \( \dim \ker(T) = 2 \) and, by the dimension theorem,

\[
\dim \text{Im}(T) + \dim \ker(T) = \dim \mathbb{F}^4.
\]

The last implies that \( \dim \text{Im}(T) = 2 \), hence it cannot equal \( \text{span}\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \} \)

which is three dimensional. \[ \square \]