INSTRUCTIONS

• Each question is worth 10 points.
• To receive full credit, you must explain your answers.
• Solutions are to be the result of an individual effort. For this examination, communication or collaboration with anyone other than the instructor is not allowed.
• Authorized materials are limited to the course textbook, class notes, and the problem sets (including solutions). Use of other resources is not allowed.
• Students are responsible for supporting and upholding the fundamental values of academic integrity.

PROBLEMS

1. Recall that for any group $G$, the quotient group $G/[G,G]$ is abelian. Let $\text{Gr}$ and $\text{Ab}$ denote the categories of groups and of abelian groups respectively.
   (a) If $G$ is a group and $A$ is an abelian group, show that there is a bijection between $\text{Hom}_{\text{Gr}}(G,A)$ and $\text{Hom}_{\text{Ab}}(G/[G,G],A)$.
   (b) The infinite dihedral group $D_\infty$ is finitely generated and hence $D_\infty/[D_\infty,D_\infty]$ is a finitely generated abelian group. Determine $D_\infty/[D_\infty,D_\infty]$.

2. Classify all groups of order 42.

3. Let $\mathbb{F}$ be a field.
   (a) Prove that $\mathbb{F}[x,y]/(x^2-y) \cong \mathbb{F}[t]$.
   (b) Prove that $\mathbb{F}[x,y]/(x^2-y) \not\cong \mathbb{F}[x,y]/(x^2-y^2)$.

4. Let $\mathbb{Z}[[x]]$ denote the ring of formal power series with integer coefficients.
   (a) Describe the units of $\mathbb{Z}[[x]]$.
   (b) Is $x^2 + 3x + 2$ irreducible in $\mathbb{Z}[[x]]$?

5. (a) Let $A \subset B$ be abelian groups. Prove that there exists a subgroup $C \subset B$ such that $B = A \oplus C$ if and only if there is a homomorphism $g : B \to A$ such that $g \circ \iota = 1_A$, where $\iota : A \to B$ is the inclusion map.
   (b) Let $A$ be a divisible abelian group, i.e., for every $a \in A$ and every nonzero $m \in \mathbb{Z}$ there exists $a' \in A$ such that $a = ma'$. Prove that, as in (a) above, whenever $A \subset B$, there exists $C \subset B$ such that $B = A \oplus C$.  

MATH 893: page 1 of 2
6. For an abelian group $A$, the Pontryagin dual is the group $D(A) := \text{Hom}_\mathbb{Z}(A, \mathbb{Q}/\mathbb{Z})$.

(a) Find $D(\mathbb{Z})$.

(b) Prove that the map $\Psi : A \to D(D(A))$ defined by

$$\Psi(a)(\theta) = \theta(a), \quad a \in A, \quad \theta \in D(A),$$

is a homomorphism of abelian groups.

(c) If $A$ is a finite abelian group, prove that $\Psi$ is an isomorphism.

*Hint:* Reduce to the case when $A = \mathbb{Z}_m$.

(d) Is $\Psi : \mathbb{Z} \to D(D(\mathbb{Z}))$ an isomorphism? Is it injective? Is it surjective?