Assignment #2
Due: 24 September 2019

1. Let \( F \) be a field and let \( \text{Vect}_F \) be the category of vector spaces over \( F \).
   (a) Prove that products and coproducts in \( \text{Vect}_F \) exist.
   (b) If \( I \) is an index set and, for every \( i \in I \), \( V_i \) is a vector space, describe
       \( \prod_{i \in I} V_i \) and \( \coprod_{i \in I} V_i \).
   (c) For what \( I \) are the vector spaces \( \prod_{i \in I} V_i \) and \( \coprod_{i \in I} V_i \) isomorphic regardless of the choice of the vector spaces \( V_i \)?

2. Let \( C \) be a category and \( Z \) be an object in \( C \). Define a new category \( C_Z \) whose objects are pairs \((X, \varphi)\), where \( X \) is an object in \( C \) and \( \varphi \in \text{Hom}_C(X, Z) \) and
   \[
   \text{Hom}_{C_Z}((X, \varphi), (Y, \psi)) = \{ \theta \in \text{Hom}_C(X, Y) \mid \psi \circ \theta = \varphi \}.
   
   The categorical product of \((X, \varphi)\) and \((Y, \psi)\) in \( C_Z \) (if it exists) is called fibre product of \( \varphi \) and \( \psi \) in \( C \) and is denoted \( X \times_Z Y \).
   (a) Prove that in the category \( \text{Sets} \) the fibre product \( X \times_Z Y \) exists and describe it.
   (b) Describe the coproduct of \((X, \varphi)\) and \((Y, \psi)\) in \( C_Z \) in \( \text{Sets} \).

   Remark. Note that the coproduct in \( C_Z \) is not called fibred coproduct. What would be called fibred coproduct?

3. (a) Let \( G \) be a group such that \( |x| = 2 \) for every \( x \neq e \). Prove that \( G \) is abelian.
   (b) Let \( G \) be an abelian group. Prove that the set of elements of \( G \) of finite order is a subgroup of \( G \).
   (c) Consider the following elements of \( \text{GL}_2(\mathbb{R}) \):
       \[
       a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
       
       Show that \( |a| = 3, |b| = 4 \), but \( |ab| = \infty \).

4. Let \( S_{n+1} \) be the symmetric group on the set \( \{1, 2, \ldots, n, n+1\} \) and let, for \( 1 \leq i \leq n \), \( s_i \) denote the transposition \((i, i+1)\).
   (a) Prove that \( s_1, s_2, \ldots, s_n \) generate \( S_{n+1} \).
   (b) Prove the relations: \( s_i^2 = e \), \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \), and \( s_i s_j = s_j s_i \) for \( |i-j| \geq 2 \).
   (c) Define the length \( \ell(\sigma) \) of \( \sigma \in S_{n+1} \) as the smallest \( k \) for which \( \sigma = s_{i_1} s_{i_2} \ldots s_{i_k} \).
      (Note that the expression itself is not unique: \( \ell(13) = 3 \) as \( (13) = (12)(23)(12) = (23)(12)(23) \).) Prove that
      \[
      \ell(\sigma) = \# \{ (i, j) \mid i < j \text{ but } \sigma(i) > \sigma(j) \}.
      
5. Let \( G \) be a finitely generated group. Prove that \( G \) admits a maximal proper subgroup.