Assignment #11
Due: 5 December 2019

1. Let $R$ be a Principal Ideal Domain. What is the minimal number of generators of the $R$-module

$$R^k \oplus R/(m_1) \oplus R/(m_2) \oplus \cdots \oplus R/(m_s) \quad \text{where} \quad 0 \neq (m_1) \subset (m_2) \subset \cdots \subset (m_s) \neq R?$$

Prove your answer!

2. Let $F$ be a subfield of $K$ and let $A$ and $B$ be $n \times n$ matrices with entries in $F$. Assume that there exists an invertible matrix $Q$ with entries in $K$ such that $B = QAQ^{-1}$. Prove that there is an invertible matrix $R$ with entries in $F$ such that $B = RAR^{-1}$.

3. (a) Let $A$ be a $3 \times 3$ matrix with rational entries such that $A^8 = I_3$. Prove that $A^4 = I_3$.
   (b) Write down a $3 \times 3$ matrix $B$ with real entries such that $B^4 \neq I_3$ but $B^8 = I_3$.

4. (a) In $F_{19}[x]$ the polynomial $x^5 - 1$ decomposes into irreducibles as $x^5 - 1 = (x - 1)(x^2 - 4x + 1)(x^2 + 5x + 1)$. Write down (up to similarity) all $2 \times 2$ matrices $A$ with entries in $F_{19}$ such that $A^5 = I_2$.
   (b) Write down representatives of the conjugacy classes in the group $GL_3(F_2)$.

5. Let $V$ be a finite dimensional vector space over the algebraically closed field $F$ and let $T : V \rightarrow V$ be a linear transformation.
   (a) Prove that $T$ can be written as $T = S + N$ where $S$ is diagonalizable, $N$ is nilpotent, and $SN = NS$.
   (b) Prove that there exists $p(x), q(x) \in F[x]$ such that $S = p(T)$ and $N = q(T)$.
   (c) Prove that the transformations $S$ and $N$ from (a) are unique.
   (d) Give an example of linear transformation of a real vector space which does not admit a decomposition as in (a).