Shifted convolutions and applications

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Number Theory by early career researchers

Motivation: Moments

- 1859: B. Riemann: RH.
- 1908: E. Lindelöf: LH. (RH $\implies$ LH; LH $\not\Longrightarrow$ RH.)

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll (1+|t|)^{\epsilon} \quad (\forall \epsilon > 0).$$

Lindelöf: exponent $1/4$ (convexity bound).

- 1916: Hardy, Littlewood (via Weyl's differencing method): Weyl's bound (exponent $1/6$).

Different approach: Via moments: Drop all but 1 term: $T < t \leq 2T$, $T \to \infty$,

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \left(\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt\right)^{\frac{1}{2k}} \; (\overset{\forall k}{\Longrightarrow} \text{ LH}).$$

- 1917: Hardy, Littlewood: 2nd moment of zeta (used AFE for $\zeta(s)$).

THEOREM A.—*Suppose that H and K are positive constants, and*

(1.11) $$s=\sigma+it, \quad -H\leqslant\sigma\leqslant H,$$

(1.12) $$x>K, \quad y>K, \quad 2\pi xy=|t|.$$

Then (i)

(1.13) $$\zeta(s)=\sum_{n<x} n^{-s}+\chi\sum_{n<y} n^{s-1}+R,$$

where

(1.14) $$\chi=2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s),$$

(1.15) $$R=O(x^{-\sigma})+O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}),$$

$$\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)|^2\,dt\sim 2T\log T.$$

This result was proved in our memoir "Contributions . . . .", *Acta Mathematica*, Vol. 41 (1917), pp. 119–196 (pp. 151–156).

- 1926: Ingham: 4th moment (used AFE for $\zeta(s)^2$).

$$\zeta^2(\tfrac{1}{2}+ti) = \sum_{1}^{t/2\pi} d(n)\, n^{-\frac{1}{2}-ti} + i\left(\frac{t}{2\pi e}\right)^{-2ti} \sum_{1}^{t/2\pi} d(n)\, n^{-\frac{1}{2}+ti} + O(\log t),$$

Theorem B. *We have, as* $T\to\infty$,

$$\int_0^T |\zeta(\tfrac{1}{2}+ti)|^4\, dt = \frac{1}{2\pi^2} T\log^4 T + O(T\log^3 T).$$

- 1979: Heath-Brown: 4th moment with power saving error term (used AFE for $|\zeta(s)|^2$).

LEMMA 1. *Let $k \geqslant 1$ be an integer. There exist constants c, $\alpha(u,v)$, and an integer U, all depending on k, such that $c \geqslant 1$ and*

$$|\zeta(\tfrac{1}{2}+it)|^{2k} = \sum_{mn\leqslant cT^k} d_k(m)\, d_k(n)(mn)^{-\frac{1}{2}}(m/n)^{it} K(mn,t) + O(T^{-2}),$$

THEOREM 1. *There exist constants a_4, a_3, a_2, a_1, a_0 such that, for $T \geqslant 2$ and $\varepsilon > 0$,*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\, dt = a_4 T(\log T)^4 + a_3 T(\log T)^3 + a_2 T(\log T)^2 + a_1 T\log T + a_0 T + O(T^{7/8+\varepsilon}). \qquad (2)$$

Shifted convolutions

Heath-Brown 1979, Thm. 2

$\exists c_0(h), \cdots, c_2(h)$ such that, uniformly for $h \leq X^{5/6}$,

$$\sum_{n \leq X} \tau_2(n)\tau_2(n+h) = \sum_{i=0}^{2} c_i(h) X \log^i X + O(X^{\frac{5}{6}+\epsilon}).$$

Precise guess for higher moments not known until

1984: Conrey, Ghosh conjectured

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^6 dt \sim 42 a_6 T \frac{(\log T)^9}{9!} \quad (T \to \infty).$$

Need full asymptotic with sharp error term for

$$\sum_{n \leq X} \tau_3(n)\tau_3(n+h), \quad (1 \leq h \leq X^{1/3}).$$

Bottle-neck

2001: Conrey, Gonek conjectured

$$\sum_{n\leq X}\tau_3(n)\tau_3(n+h)=m_3(X,h)+O(X^{1/2+\epsilon}),\ (1\leq h\leq X^{1/2}).$$

2016: N. Ng: Smoothed ternary shifted convolutions $\Longrightarrow$ full asymptotic for 6th moment of zeta (with power-saving error term).
2022: I provided numerical evidence for a special case of Conrey–Gonek's conjecture [arXiv:2206.05877].
Find/compute five numerical constants $b_0,\ldots,b_4$ such that

$$\sum_{n\leq X}\tau_3(n)\tau_3(n+1)-X\left(b_4\log^4 X+b_3\log^3 X+\cdots+b_0\right)\leq C(\epsilon)X^{\frac{1}{2}+\epsilon}. \qquad (1)$$

I numerically verify (1) for $X\leq 10^6$ (and $\epsilon=0.01$).

$$b_4 = 0.054444679154884094580751878529861703282699438750338984412069100880906622778063155 13$$

$$b_3 = 0.71011392905364474755395892667350537295819711946375750493984571 5$$

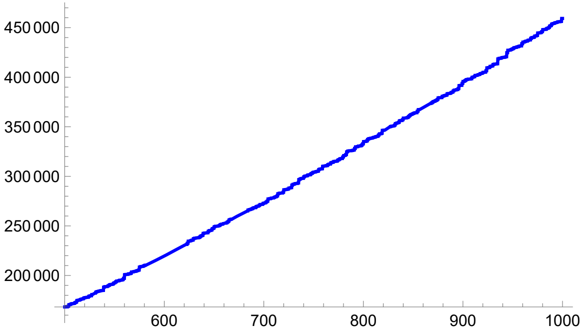
$$b_2 = 2.02119605787987777943324240784753809467091508369917789267040603 5$$

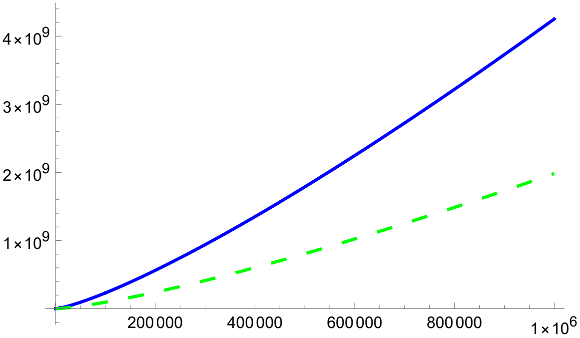
$$b_1 = 0.67786331083298038854157108306273365600322232270413534868810242 51$$

$$b_0 = 0.28723664774661941722166461781464595016603627439722224961891390 74.$$

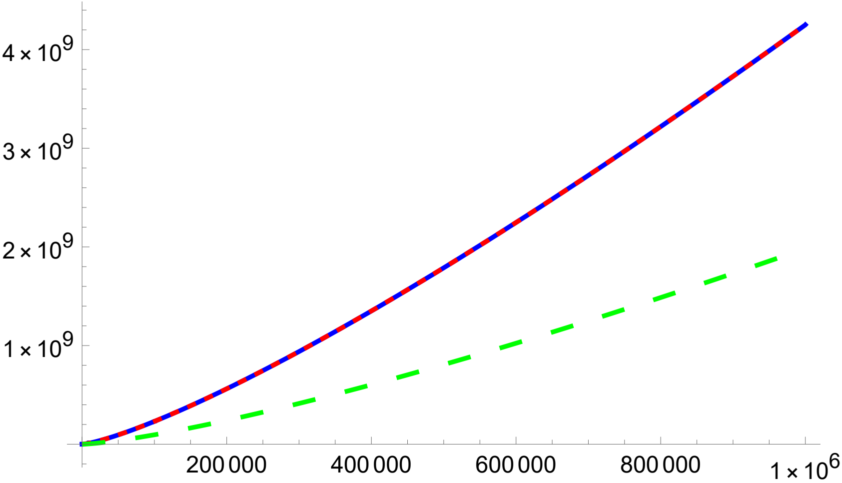
Mathematica file: `https://aimath.org/~dtn/papers/correlations/Proof_of_Cor_1.nb` or `https://github.com/nguyen-d-8/correlations`

- solid blue is data $\sum_{n\le X}\tau_3(n)\tau_3(n+1)$
- dashed green is leading order term $\frac{1}{4}\prod_p\left(1-\frac{4}{p^2}+\frac{4}{p^3}-\frac{1}{p^4}\right)X\log^4 X$



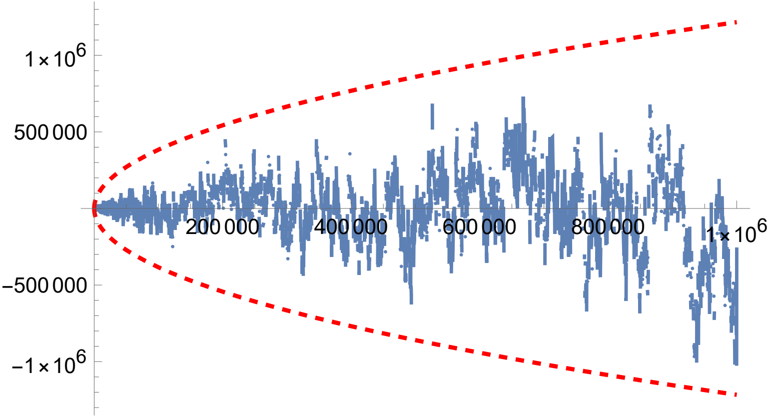


- dotted red is prediction
- solid blue is data $\sum_{n\leq X} \tau_3(n)\tau_3(n+1)$

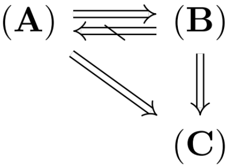


What about the error term?

- jagged blue is the error: $\sum_{n \le X} \tau_3(n)\tau_3(n+1) -$ prediction.
- dashed red is the bounds $\pm 1050 X^{0.51}$



I believe $\sum_{n\leq X} \tau_3(n)\tau_3(n+h)$ is "too strong" for certain applications.
Suppose (A) and (C) are two statements, possibly conjectures, with (A) $\implies$ (C). We say that (A) is "too strong" for (C) if there exists a statement (B) such that, (i), (B) is easier to prove than (A), and, (ii), the following diagram of implications holds:



I propose to study the following modified weaker shifted convolution

$$\sum_{n\leq X-h} \tau_3(n)\tau_3(n+h). \tag{2}$$

I showed that this sum (2) is close to its expected value in an L^2 sense, and that this is enough for certain problems.

I proved, with a power-saving error term, that the second moment of (2) is small:

$$\sum_{h<X}\left(\sum_{n\le X-h}\tau_3(n)\tau_3(n+h)-\mathrm{MT}(X,h)\right)^2 \ll X^{3-1/100}. \qquad \textbf{(B)}$$

As an application of the above bound (**B**), we obtain the full asymptotic for the variance of the ternary 3-fold divisor function in arithmetic progressions, averaged over all residue classes (not necessarily coprime) and moduli: There exist computable numerical constants $c_0,\ldots,c_8$ such that

$$\sum_{q\le X}\sum_{1\le a\le q}\left(\sum_{\substack{n\le X\\ n\equiv a(q)}}\tau_3(n)-\mathrm{MT}(X;q,a)\right)^2 = X^2(c_8\log^8 X+\cdots+c_0)+O\left(X^{2-1/300}\right), \qquad \textbf{(C)}$$

for some explicit main term $\mathrm{MT}(X;q,a)$.
[arXiv:2302.12815] (To appear: *Proc. Edinb. Math. Soc.*)

- Quantities of the form (**C**) are of Barban–Davenport–Halberstam type and have their roots in the celebrated Bombieri-Vinogradov Theorem.
- Barban–Davenport–Halberstam type inequalities have many applications in number theory. For instance, a version of this inequality (with $\Lambda(n)$ replaced by related convolutions over primes) was skillfully used by Yitang Zhang (2014) in his spectacular work on bounded gaps between primes.

Lemma 10. *Suppose that $\beta = (\beta(n))$ satisfies (A_2) and $R \leq x^{-\varepsilon}N$. Then for any q we have*

$$\sum_{r\sim R} \varrho_2(r) \sum_{l(\bmod r)}^{*} \Bigg| \sum_{\substack{n\equiv l(r)\\(n,q)=1}} \beta(n) - \frac{1}{\varphi(r)} \sum_{(n,qr)=1} \beta(n) \Bigg|^2 \ll \tau(q)^B N^2 \mathcal{L}^{-100A}.$$

- In their work in 2017, Heath-Brown and Li proved a version of Barban–Davenport–Halberstam inequality in their Corollary 2 as an ingredient to show that the sparse sequence $a^2 + p^4$, where a is a natural number and p is a prime, contains infinitely many primes.

Theorem 2 *For any $Q \in \mathbb{N}$, let*

$$\mathcal{E}(N_1, N_2, Q_0) = \sum_{q\leq Q} \sum_{a\,(\bmod q)}^{*} |\mathcal{E}(a, q; N_1, N_2, Q_0)|^2.$$

Then,

$$\mathcal{E}(N_1, N_2, Q_0) \ll \left(Q + \frac{N_1 N_2}{Q_0}\right)(\log Q)\|\gamma\tau^{1/2}\|^2\|\delta\tau^{1/2}\|^2.$$