## 2023-2024 – McMaster University Algebra and Algebraic Geometry Seminar (McMaster AAGS)

Date: Thursday 16 November 2023

Time: 9:30–10:20AM EDT

Location: Hamilton Hall 207

Speaker: David Nguyen (Queen's)

Title: An improved upper bound on a class of exponential sums.

Abstract: In Yitang Zhang's proof of bounded gaps between primes (2014), a certain 3-variable Kloosterman sum played a crucial role and was one of the deepest parts of his proof. Further improvements on bounded gaps will likely require deeper understanding of this sum. This particular exponential sum was originally studied by J. Friedlander and H. Iwaniec (1985) with upper bound first obtained by B. Birch and E. Bombieri (1985) by special arguments. Further improvements were made by N. Katz (1986) using $\ell$-adic method, and very recently by C. Chen and X. Lin (2022) by p-adic method. In an attempt to understand a remark made by Katz at the end of his 1986 paper, M. Roth and I were able to find a way that leads to a slight improvement on the upper bound of this sum, as well as on a family of similar exponential sums. Our method is based on $\ell$-adic cohomology, consisting of finer studying of the local monodromies at zero via representations of the inertia groups there. In this talk, I will survey the above-mentioned results and give a high-level overview of our procedure that gives rise to this new improvement. This is joint work in progress with M. Roth.

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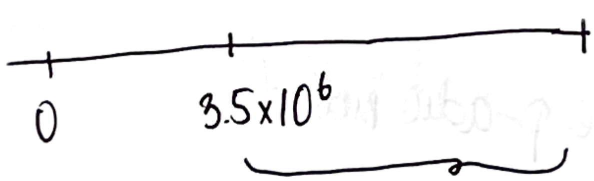
## Exponential Sums

- **Aim**: Reduce bound on gaps b/w primes from 246 → 244, say.
- (Def$^n$) $H = \{h_1, \dots, h_k\} \subseteq \mathbb{N} \cup \{0\}$ is called **admissible** if $\forall$ prime $p$ $\exists n$ s.t. $GCD\left(\prod_{i=1}^{k}(h_i+n), p\right) = 1$.
- **Ex**: $H = \{2,3\}$. No ($p=2$).
  - $H = \{0,2,4\}$ - No ($p=3$).
  - $H = \{0, 2\}$. Yes.
  - $H = \{0, 2, 6\}$ Yes.

**Conj**:
- (Hardy–Littlewood) $k$-tuple Conj. $H$ admissible $\Rightarrow \exists$ $\infty$'ly many $n$ st. $\{h_1+n, h_2+n, \dots, h_k+n\}$ (*) are all primes.
- **Thm** (Zhang) Assume $H$ is admissible w/ $k \geq 3.5\times10^6$. **Then** $\exists$ $\infty$'ly many $n$ s.t. (*) contains at least 2 primes.
- Taking ~~the~~ $H = \{k < h_1 < \dots < h_k\}$ all primes.




$$\pi(7\times10^7) - \pi(3.5\times10^6) > 3.5\times10^6 \text{ primes}.$$

- Consequently,

$$\liminf_{n\to\infty} p_{n+1} - p_n \le 7\times 10^7 - 3.5\times 10^6 \quad (\text{Zhang})$$
$$\le 246 \quad (\text{Polymath } 8)$$

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- How?
- Crucial input: For $\alpha_1\alpha_2 \neq 0$, $\alpha_1, \alpha_2 \in \mathbb{F}_p$,

$$S_p = \sum_{\substack{x_1,\dots,x_4 \in \mathbb{F}_p^\times \\ \frac{\alpha_1}{x_1x_2} + \frac{\alpha_2}{x_3x_4} = 1}} e^{\frac{2\pi i}{p}(x_1+\dots+x_4)} \ll p^3 \ (\text{trivial bound})$$
$$\ll p^2 \ (\text{by Weil 'bd})$$
$$\overset{\text{WANT}}{\ll} p^{\frac{3}{2}}$$

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- Birch-Bombieri ('85): $\exists\, C_0, C_1 > 0$ s.t. $\forall p \geq C_0$,

$|S| \le C_1 p^{\frac{3}{2}}$. (Using Deligne's Weil I)

- Katz (1986): $C_0 = 1$, $C_1 = 8$ are admissible, i.e., $\forall p$,

$|S| \le 8p^{\frac{3}{2}}$. (Used ~~Del~~ Weil II)

- Chen, Lin (2022) $\forall p$

$|S| \le 6p^{\frac{3}{2}} + p + 1$ (Use p-adic method)

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(- S is a special case of:

$S_p = \sum$ )

③

– C-L: $S_n$ exp-sums / $\mathbb{F}_{p^n}$ $(S_1 = S)$

$$\exp\left(\sum_{n=1}^{\infty} S_n \frac{T^n}{n}\right) \underset{=}{\text{Thm.}} (1-T)(1-pT)\prod_{i=1}^{6}(1-a_i T), \quad |a_i| = p^{\frac{3}{2}} \ (\forall i).$$

– B-B: Worked w/ a surface $(n=2)$ < non-singular: Deligne; Abhyankar's res. of surface sing's

After reduction $n=2$

| | dim | \|Frob\| |
|---|---|---|
| $H^0$ | 1 | 1 |
| $H^1$ | 0 | $p^{\frac{1}{2}}$ |
| $H^2$ | * | $p$ |
| $H^3$ | 0 | $p^{\frac{3}{2}}$ |
| $H^4$ | 1 | $p^2$ |

Albanese map 0

- $h^3 = h^1 = 2\cdot$irregularity
- Hooley: $p \gg C_0 \Rightarrow h^4 = 0$.

B-B: $H^4(S, \mathbb{Q}_\ell)$ (wght. 0) Weil I ✓

Katz: $H^2(C, \mathcal{F})$ (wght. n) Need Weil II

– Katz: $n \geq 1$, $r \geq 1$, (consider partition)

$n = n_1 + n_2 + \dots + n_r$ $(n_i \geq 1)$

$x_{i,j}$, $j = 1, \dots, n_i$

~~$b_{i,j} \geq 1$~~

$\psi$ non-trivial add char. of $\mathbb{F}_p$.

$\alpha_1, \dots, \alpha_r \in \mathbb{F}_p^{\times}$

$$V_\beta: \ \beta = \sum_{i=1}^{r} \frac{\alpha_i}{\prod_{j=1}^{n_i}(x_{ij})^{\cancel{b_{ij}}}}$$

$$\boxed{S_\beta} := \sum_{x \in V_\beta(\mathbb{F}_p)} \psi\left(\sum_{i,j} x_{i,j}\right). \quad (\text{So } S = S_1 \text{ w/ } r=2, n_1 = n_2 = 2)$$

- ~~Kea~~ Thm (Katz) For $p \neq 0$, $|S_p| \le \left(\prod_{i=1}^{r}(n_i+1)-1\right) p^{\frac{n-1}{2}}$. ( $|S| \le 8p^{\frac{3}{2}}$ )

$m=2$

- ~~Question~~: $r=2$, $n_1=n_2=m$, $n=2m$

Katz: $|S_p| \le ((m+1)^2-1)\, p^{m-\frac{1}{2}} = (m^2+2m)\, p^{m-\frac{1}{2}}$ $\quad |S| \le 8p^{\frac{3}{2}}$

CLH: $|S_p| \le (m^2+2m-2)\, p^{m-\frac{1}{2}} + O(p^{m-1})$ $\quad |S| \le 6p^{\frac{3}{2}}$

w/ M. Roth: $|S_p| \le (m^2+m-1)\, p^{m-\frac{1}{2}} + O(p^{m-1})$ $\quad |S| \le 5p^{\frac{3}{2}}$

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(Def$^n$) A lisse sheaf $\mathcal{F}$ on $U$ "is" a continuous rep. of $\pi_1(U)$ on some $\ell$-adic ($\ell \neq p$) vector space $V$.

(étale fund. gr.)

- What is the sheaf? Katz: $\mathcal{F}$ on $\mathbb{A}^1$

Katz's insight: Take the Fourier transf. of $S_p$!

$$pS_p = \sum_{t \in \mathbb{F}_p} \psi(-tp) \prod_{i=1}^{r} Kl(i, \alpha_i t)$$

hyper-Kl sums $\displaystyle\sum_{x_{i,1},\dots,x_{i,n_i} \in \mathbb{F}_p^\times} \psi\Big(\sum_j x_{ij} + \frac{\alpha_i t}{\prod_j x_{ij}}\Big)$

- $\ell \neq p$, $\ell$-adic place $\lambda$ of $E=\mathbb{Q}(\zeta_p)$. $\exists$ lisse $E_\lambda$-sheaf on $\mathbb{G}_m$ $Kl(i)$ s.t. $\forall t \in \mathbb{A}^1$,

$$\mathrm{tr}(\mathrm{Frob}_t \mid (j_* Kl(i))_t) = (-1)^{n_i} Kl(i,t) \qquad j: \mathbb{G}_m \hookrightarrow \mathbb{A}^1.$$

$$\begin{array}{ccccc} & & \mathbb{A}^1 & \dashrightarrow & T_{\alpha_i}^* j_* Kl(i) \\ & & \downarrow{\scriptstyle T_{\alpha_i}: x \mapsto \alpha_i x} & & \mathcal{L}_{\psi(-px)} \\ \mathbb{G}_m & \overset{j}{\hookrightarrow} & \mathbb{A}^1 & & \\ | & & | & & \\ Kl(i) & & j_* Kl(i) & & \end{array}$$

$$\mathcal{F} = \bigotimes_{i=1}^{r} T_{\alpha_i}^{*} j_* Kl(\ell_i)$$

$$\therefore\ pS_p \overset{\text{def}}{=} \sum_{t\in\mathbb{F}_p} (-1)^n \operatorname{tr}(\mathrm{Frob}_t \mid (\mathcal{F}\otimes\mathcal{L})_{\bar t})$$

$$\underset{=}{\text{Lefschetz T.F.}}\ (-1)^n \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\mathrm{Frob}_p \mid H_c^i(\mathbb{A}^1\otimes\overline{\mathbb{F}}_p, \mathcal{F}\otimes\mathcal{L}))$$

$$\underset{=}{H_c^0 = H_c^2 = 0}\ (-1)^{n+1} \operatorname{tr}(\mathrm{Frob}_p \mid H_c^1(\mathbb{A}^1, \mathcal{F}\otimes\mathcal{L}))$$

$\mathcal{F}\otimes\mathcal{L}$: pure of wght. $n$

$\longrightarrow$ Deligne $\Rightarrow$ mixed $\leq n+1$

$$\underset{\leq}{\text{Deligne}}\ h_c^1 \cdot p^{\frac{n+1}{2}}.$$

$h_c^1$: $\prod_{i=1}^{r}(n_i+1) - 1$

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- $X$-smooth prj., $U \overset{j}{\hookrightarrow} X$, $S = X\setminus U$, $\mathcal{F}$ lisse on $U$ of wght. $w$
- On $X$, (one has the s.e.s)

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to \bigoplus_{s\in S} (j_*\mathcal{F})_s \to 0$$

gives rise to $\qquad$ ~~$\leq w$~~ $= w$ $\qquad$ $\leq w$ [Weil II, lemma 1.8.1]

$$0 \to H^0(X, j_*\mathcal{F}) \hookrightarrow \bigoplus_{s\in S} H^0(X, (j_*\mathcal{F})_s)$$

$$\to \boxed{H_c^1(U,\mathcal{F})} \twoheadrightarrow H^1(X, j_*\mathcal{F}) \to 0$$

$H_c^1(U,\mathcal{F}) = H^1(X, j_!\mathcal{F})$ $\qquad$ $H^1(X, j_*\mathcal{F})$: $\leq w+1$

$H^1(X, j_!\mathcal{F})$: $\leq w+1$ $\qquad$ $H^1(X, j_*\mathcal{F})$: ~~pure~~ of wgh. $w+1$ [Théorème 2 in Weil II.]

$$\begin{tikzcd}
 & j_i^*\mathcal{L} \arrow[d, dash] & \mathcal{L} \arrow[d, dash] \\
T_{\alpha_i}^* Kl(i) \arrow[r, dash] & \mathbb{G}_m \arrow[r, hook, "j_i"] \arrow[d, "T_{\alpha_i}"] & \mathbb{A}^1 \\
 & \mathbb{G}_m \arrow[d, dash] & \\
 & Kl(i) &
\end{tikzcd}$$

— $\mathcal{F}$ on $\mathbb{G}_m$:

$$\boxed{\mathcal{F}} = \left(\bigotimes_{i=1}^{r} T_{\alpha_i}^* Kl(i)\right) \otimes j^* \mathcal{L}_{\psi(-px)}$$

(lisse, pure of wght. $n$)

— $s=\infty$: $(j_*\mathcal{F})_\infty = 0$ (b/c wild ramification)

— $s=0$: weights of $(j_*\mathcal{F})_0$ are the same as weights of $\left(\bigotimes_{i=1}^{r} Kl(i)\right)_0$

" " $\bigotimes_{i=1}^{r} V_{m_i} \underset{=}{\text{Clebsch-Gordan}} \bigoplus_{i=1}^{N_r} c_i V_{\tilde{m}_i}$

↖ unipotent of dim. $m_i+1$

— Where does the improvement come from?

↳ $V_m \otimes V_m = \underbrace{V_{2m} \oplus V_{2m-2} \oplus \cdots \oplus V_0}_{m+1}$

$= 0 \quad = 2 \quad = n$

[Fr: Weil II, Thm 1.8.4] weight of $V_i = w - i$.