STRATIFICATION OF WEIGHTS ON CURVES

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ABSTRACT. We give a stratification, i.e., classification of Frobenius eigenvalues according to their sizes and multiplicities, of weights of certain sheaves, arising from analytic number theory, on the curve $\mathbb{G}_m$. As an application, we sharpen upper bounds on resulting exponential sums.

1. Introduction

This paper grows out of, and refines, a remark made by N. Katz at the end of his article [Kat87, p. 17], which says

Remark 1 (Katz). *Except in some very special and atypical cases (e.g., $r = 1$), this sheaf will not be pure of weight $n + 1$.*

Our main result (Theorem 1) gives precisely the weights that occur as well as their multiplicities, i.e., how many there are of each weight. Our interest in these fine details stem from connection to the problem of reducing gaps between consecutive primes.

Denote by $V_{\beta,\vec{\alpha},p}(r; n_1, \ldots, n_r)$ the subvariety of $(\mathbb{G}_m \times \mathbb{F}_p)^{n_1+\cdots+n_r}$ (with coordinates $x_{j,k}$) defined by the equation $\sum_{j=1}^r \alpha_j \prod_{k=1}^{n_j} \overline{(x_{j,k})^{b_{j,k}}} = \beta$, where $r \geq 1$, $b_{j,k} \geq 1$, $\beta \in \mathbb{F}_p^\times$, $\vec{\alpha} = (\alpha_j)_{j=1}^r$, $\alpha_j \in \mathbb{F}_p$ $(\forall j)$, $n_j \geq 1$ $(\forall j)$, and $\bar{\cdot}$ denotes multiplicative inverse modulo p. We form the following exponential sum, which constitute the main object of the paper,

$$S_\beta = S_{\beta,\vec{\alpha},p}(r; n_1, \ldots, n_r) = \sum_{x \in V_{\beta,\vec{\alpha},p}(r;n_1,\ldots,n_r)} \exp\left(\frac{2\pi i}{p} \sum_{\substack{1\leq j\leq r \\ 1\leq k\leq n_j}} x_{j,k}\right). \tag{1.1}$$

For ease of exposition, assume in this section that all $b_{j,k} = 1$. A result of Katz [Kat87] implies

$$|S_{\beta,\vec{\alpha},p}(r; n_1, \ldots, n_r)| \leq C(r; n_1, \ldots, n_r) p^{\frac{n_1+\cdots+n_r-1}{2}} \tag{1.2}$$

for all p, and for some explicit constant $C(r; n_1, \ldots, n_r)$ independent of β, $\vec{\alpha}$, and, more crucially, p. Katz gave crudely that $C(r; n_1, \ldots, n_r) = (\prod_{i=1}^r (1 + n_i)) - 1$ is admissible. His Remark 1 then, in one interpretation, translates to the strict inequality that

$$\lim_{p\to\infty} \frac{|S_{\beta,\vec{\alpha},p}(r; n_1, \ldots, n_r)|}{p^{\frac{n_1+\cdots+n_r-1}{2}}} < \left(\prod_{i=1}^r (1 + n_i)\right) - 1, \tag{1.3}$$

unless $r = 1$ (in which case one gets equality). Subsequent improvements to the bound (1.3) were made by C. Chen and X. Lin [CL22a; CL22b]–see Section 1.1 for more. Our result gives improvement to the right side of (1.3) in both the n_i and r aspects.

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Corollary 1 (Number-of-variables n_i aspect)**.** *We have, for $r = 2$ and $n_1 = n_2 = m \geq 1$,*

$$\lim_{p\to\infty} \frac{|S_{1,\vec{\alpha},p}(2; m, m)|}{p^{m-\frac{1}{2}}} = m^2 + m$$

for all $\alpha_1, \alpha_2 \in \mathbb{F}_p$ with $\alpha_1\alpha_2 \neq 0$. We note that $\prod_{i=1}^{2}(1 + m) - 1 = m^2 + 2m$.

Corollary 2 (Number-of-summands r-aspect)**.** *We have, as $r \to \infty$ and $n_1 = \cdots = n_r = 2$,*

$$\lim_{p\to\infty} \frac{|S_{\beta,\vec{\alpha},p}(r; 2, \ldots, 2)|}{p^{r-\frac{1}{2}}} \leq 3^r \left(1 - \sqrt{\frac{3}{\pi}} r^{-1/2}\right)$$

for all $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_p$ with $\alpha_1 \cdots \alpha_r \neq 0$. We note that $\left(\prod_{i=1}^{r}(1 + 2)\right) - 1 = 3^r - 1$.

These two corollaries follow from the more general Theorem 1 below. We divide into these two cases for ease of comparison with previous results; see Table 1 below.

TABLE 1. Comparisons of the leading constant $C(r; n_1, \ldots, n_r)$ in Equation (1.2). The smaller the sharper the result.

$C(r; n_1, \ldots, n_r)$	Katz [Kat87]	Chen, Lin [CL22a; CL22b]	Corollaries 1 and 2
$C(r; 2, \ldots, 2)$	$3^r - 1$	$3^r - 1 - c_1 r^2$	$3^r - c_2 3^r r^{-1/2}$
$C(2; m, m)$	$m^2 + 2m$	$m^2 + 2m - 2$	$m^2 + m$

1.1. **Survey of previous results.** Let notation be as above. The sum $S_{1,\vec{\alpha},p}(2; 2, 2)$ was first studied by J. Friedlander and H. Iwaniec in [FI85] in the context of beating the generalized Riemann hypothesis. About three decades later, this exact sum $S_{1,\vec{\alpha},p}(2; 2, 2)$ reappeared (through an elaborate series of reductions) as a crucial ingredient to Y. Zhang's proof [Zha14] in the different setting of bounded gaps between primes (see [Zha14, Lemma 12, p. 1135]). The bound needed for this sum in both of these applications is $|S_{1,\vec{\alpha},p}(2; 2, 2)| \ll p^{3/2}$. Deligne's Riemann hypothesis for curves (or Weil's bound) directly can only give $|S_{1,\vec{\alpha},p}(2; 2, 2)| \ll p^2$.

B. Birch and E. Bombieri [BB85] were the first to obtain, in the Appendix to [FI85], that

$$|S_{1,\vec{\alpha},p}(2; 2, 2)| \leq c_1 p^{3/2},$$

for all $p \geq c_0$, uniformly for any $\alpha_1, \alpha_2 \in \mathbb{F}_p$ with $\alpha_1\alpha_2 \neq 0$, where c_0, c_1 are absolute positive constants. Birch and Bombieri used Deligne's Weil I [Del73] and worked with a constant sheaf pure of weight 0 over a surface. Because of this, their proof relied on resolution of surface singularity in positive characteristics, which made it challenging to generalize, and also does not work for all p.

Through an insightful observation, Katz in [Kat87] fixed these issues. He took the Fourier transform. This converts the problem to one of analyzing certain sheaves on a curve but of higher weights. Therefore, Katz relied on Weil II [Del80]. He worked out explicitly the implied constant for a more general class of exponential sums that includes $S_{\beta,\vec{\alpha},p}$ as a special case. A specialization of [Kat87, Theorem, p. 13] gives

$$|S_{1,\vec{\alpha},p}(2; 2, 2)| \leq 8p^{3/2}, \tag{1.4}$$

and, a bit more generally,

$$|S_{\beta,\vec{\alpha},p}(2;m,m)| \le (m^2+2m)p^{m-\frac{1}{2}} \tag{1.5}$$

for all p, uniformly for any $\beta \neq 0$, $\alpha_1, \alpha_2 \in \mathbb{F}_p$ with $\alpha_1\alpha_2 \neq 0$.

By a completely different method, the constant 8 in Equation (1.4) and m^2+2m in Equation (1.5) were recently improved by C. Chen and X. Lin in [CL22a, Corollary 1.2.] and [CL22b], to 6 and m^2+2m-2, respectively. Chen and Lin used p-adic methods which, in particular, include Adolphson–Sperber's work on toric exponential sums and D. Wan's decomposition theorems. They worked with the generating function of the exponential sums over all field extensions and determined the weights and slopes of the resulting L-functions attached to these sums.

We now briefly comment on our method of proof, and indicate how our approach differs.

1.2. **Outline of the proofs.** Our method uses ℓ-adic methods–so is similar to Katz's–the difference however is that our analysis are more detailed and simplified. Specifically, we define a sheaf on $\mathbb{G}_m$ and analyzed the local monodromy at missing point zero via representations of the inertial group there.

A standard way of analyzing sheaves on curves is to prove that these sheaves, when viewed as representations of π_1, are absolutely irreducible. This implies that $\mathrm{H}^2_c = 0$ since, up to a Tate twist, the H^2_c are the coinvariants of these representations. Absolutely irreducibilities are, however, often difficult to prove. We go around this difficulty by constructing a sheaf that is lisse on $\mathbb{G}_m$ and totally wild at ∞, and exploit the fact that the H^2_c vanishes due to wildness (see Lemma 3). This modification simplifies our proof.

Our detailed analysis is stimulated by another remark of Katz but in another article [Kat01]. He wrote on page 130 there that "the weight drops reflect the structure of the local monodromies at the missing points" and that "to prove these finer results requires Deligne's detailed analysis [Del80, Section 1.8.4] of local monodromy on curves, and of its interplay with weights." We carry this out in our paper.

The proof of Theorem 1 is broken into three steps.

Step 1. Construction of the corresponding sheaf. Choose $\mathcal{F}$ and U so that, via the Grothendieck–Lefschetz trace formula, the original sum turns into an alternating sum of Frobenius eigenvalues of the shape

$$S_\beta = \frac{(-1)^n}{p}\left(\sum_{i=0}^{2}(-1)^i \mathrm{tr}\left(\mathrm{Frob} \ \mid\ \mathrm{H}^i_c\left(U,\mathcal{F}\right)\right)+1\right).$$

Step 2. Vanishing of H^2_c. If the $\mathcal{F}$ is above is also lisse on U and totally wild at a point outside of U, then one gets for free, by the wild vanishing Lemma 3 below, that $\mathrm{H}^2_c\left(U,\mathcal{F}\right)=0$.

Step 3. Stratification of weights in H^1_c. We now apply the stratification Proposition 1, which classifies and relates the weights of Frobenius eigenvalues on $\mathrm{H}^1_c(U,\mathcal{F})$ to those of the stalks of the push-forward of $\mathcal{F}$ at the remaining missing points. Via representation of the inertial group at these remaining missing points, the weights on these stalks can be counted exactly by a combinatorics lemma that we develop; see the multi Clebsch–Gordan Lemma 10 below.

Corollaries 1 and 2 follow from Theorem 1 by specializing r and n_i into the leading constant given by Equation (4.1), and applying some standard asymptotic analysis from Lemma 11.

2. Lemmas

2.1. **Algebraic lemmas.**

Lemma 1. *Let X be a smooth projective curve over $\mathbb{F}_p$, and $\mathcal{F}$ a sheaf on X. If $\mathcal{F}$ is mixed of weight $\leq w$, then $\mathrm{H}^i(X, \mathcal{F})$ is mixed of weight $\leq i + w$.*

Proof. This follows from Deligne's fundamental Théoremè 3.3.1, p. 204, in [Del80]. □

Lemma 2 (Weights of push forward)**.** *If $\mathcal{F}_0$ is a lisse sheaf over U_0, pointwise ι-pure of weight β, then, for each $x \in |S_0|$ and every eigenvalue α of F_x on $j_*\mathcal{F}_0$, we have $w_{\mathrm{N}(x)}(\alpha) \leq \beta$.*

Proof. This is Lemme 1.8.1, p. 175, of [Del80]. □

Theorem A (Purity of pushfowards)**.** *Let X_0 be a proper and smooth curve over $\mathbb{F}_q$, $j : U_0 \to X_0$ the inclusion of a dense open set, and $\mathcal{F}_0$ a point-wise pure lisse sheaf of weight n on U_0. Then, $\mathrm{H}^i(X, j_*\mathcal{F})$ is pure of weight $n + i$.*

Proof. This is Théoremè 3.2.3, p. 200, of [Del80]. □

Theorem B (Weight drop)**.** *If $\mathcal{F}_0$ is pointwise ι-pure of weight β, then the representation $\mathrm{Gr}_i^M(\mathcal{F}_{0\eta})$ of $W(\overline{\eta}/\eta)$ is ι-pure of weight $\beta + i$.*

Proof. This is Théoremè 1.8.4, p. 175, of [Del80]. □

Lemma 3 (Wild vanishing)**.** *Let X be a smooth projective curve over $\mathbb{F}_p$, $U \subsetneq X$ a nonempty open set, and $\mathcal{F}$ a lisse sheaf on U. If $\mathcal{F}$ is totally wild at some $x \in X - U$, then $\mathrm{H}^2_c(U, \mathcal{F}) = 0$.*

Proof. This is a special case of [Kat88, Lemma 2.1.1 (1), p. 28]. □

Lemma 4. *The sheaf $\mathcal{F}$ on $\mathbb{G}_m$ defined by Equation (4.3) is lisse of rank*

$$\prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} \frac{b_{i,j}}{\mathrm{GCD}(b_{i,j}, p)}\right)$$

pure of weight $n = \sum_{i=1}^{r} n_i$, and tame at zero.

Proof. This follows from the fact that each $\mathrm{Kl}(i)$ on $\mathbb{G}_m$ is lisse of rank

$$1 + \sum_{j=1}^{n_i} \frac{b_{i,j}}{\mathrm{GCD}(b_{i,j}, p)},$$

pure of weight n_i, and tame at zero (cf. [Kat88, p. 4.1.1], [Kat87, Lemma 1, p. 14]). □

Lemma 5. *Let $\mathcal{F}$ be the sheaf on $\mathbb{G}_m$ defined by Equation (4.3). We have*

$$\chi_c(\mathbb{G}_m, \mathcal{F}) = -\prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} \frac{b_{i,j}}{\mathrm{GCD}(b_{i,j}, p)}\right). \tag{2.1}$$

Proof. For $\beta \neq 0$, every ∞-break of $\mathcal{F}$ is 1. Thus,

$$\chi_c(\mathbb{G}_m, \mathcal{F}) = -\mathrm{Swan}_\infty(\mathcal{F}) = -\mathrm{rank}(\mathcal{F}).$$

Equation (2.1) then follows immediately from Lemma 4. □

2.2. **Number theory lemmas.**

Lemma 6. *Let* S_β *and* $\mathrm{Kloos}(i,a)$ *be defined as in Equations* (1.1) *and* (4.2)*, respectively. Then, for any* $t \in \mathbb{F}_p$*, we have*

$$\sum_{\beta\in\mathbb{F}_p} \psi(t\beta) S_\beta = \prod_{i=1}^{r} \mathrm{Kloos}(i, \alpha_i t). \tag{2.2}$$

Proof. The proof is elementary and makes use of the orthogonality condition of additive characters– we include it here for completeness. For $\gamma, \gamma' \in \mathbb{F}_q$, we have

$$\frac{1}{p}\sum_{\beta\in\mathbb{F}_p} \psi(\beta(\gamma-\gamma')) = \begin{cases} 1, & \text{if } \gamma = \gamma', \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Using Equation (2.3), we detect the condition

$$\sum_{i=1}^{r} \frac{\alpha_i}{\prod_{j=1}^{n_i} (x_{i,j})^{b_{i,j}}} = \beta$$

by

$$\frac{1}{p}\sum_{\beta_1\in\mathbb{F}_p} \psi\left(\beta_1\left(\sum_{i=1}^{r} \frac{\alpha_i}{\prod_{j=1}^{n_i} (x_{i,j})^{b_{i,j}}} - \beta\right)\right).$$

Thus, using the above in the definition of S_β, we have

$$\sum_{\beta\in\mathbb{F}_p} \psi(t\beta) S_\beta = \sum_{\beta\in\mathbb{F}_p} \psi(t\beta) \sum_{(x_{i,j})^{b_{i,j}}\in\mathbb{F}_p^\times} \psi\left(\sum_{i,j} (x_{i,j})^{b_{i,j}}\right) \frac{1}{p}\sum_{\beta_1\in\mathbb{F}_p} \psi\left(\beta_1\left(\sum_{i=1}^{r} \frac{\alpha_i}{\prod_{j=1}^{n_i} (x_{i,j})^{b_{i,j}}} - \beta\right)\right).$$

Bringing the β sum inside and evaluate this sum using Eq. (2.3) once more, the above is equal to

$$\sum_{(x_{i,j})^{b_{i,j}}\in\mathbb{F}_p^\times} \psi\left(\sum_{i,j} (x_{i,j})^{b_{i,j}}\right) \sum_{\beta_1\in\mathbb{F}_p} \psi\left(\sum_{i=1}^{r} \frac{\alpha_i\beta_1}{\prod_{j=1}^{n_i} (x_{i,j})^{b_{i,j}}}\right) \frac{1}{p}\sum_{\beta\in\mathbb{F}_p} \psi(\beta(t-\beta_1))$$
$$= \sum_{(x_{i,j})^{b_{i,j}}\in\mathbb{F}_p^\times} \psi\left(\sum_{i,j} (x_{i,j})^{b_{i,j}} + \sum_{i=1}^{r} \frac{\alpha_i t}{\prod_{j=1}^{n_i} (x_{i,j})^{b_{i,j}}}\right).$$

Since ψ is additive, the above in turns can be written as

$$\left(\sum_{x_{1,j}\in\mathbb{F}_p^\times} \psi\left(\sum_j (x_{1j})^{b_{i,j}} + \frac{\alpha_1 t}{\prod_j x_{1,j}}\right)\right)\left(\sum_{x_{2,j}\in\mathbb{F}_p^\times} \psi\left(\sum_j (x_{2,j})^{b_{i,j}} + \frac{\alpha_1 t}{\prod_j x_{2j}}\right)\right)\times$$
$$\cdots \times \left(\sum_{x_{r,j}\in\mathbb{F}_p^\times} \psi\left(\sum_j x_{rj} + \frac{\alpha_1 t}{\prod_j (x_{r,j})^{b_{i,j}}}\right)\right) = \prod_{i=1}^{r} \mathrm{Kloos}(i, \alpha_i t),$$

which is the right side of Equation (2.2). □

By Fourier inversion, we obtain the following

Lemma 7 (Discrete Fourier inversion)**.** *Let* S_β *and* $\mathrm{Kloos}(i,a)$ *be defined as in Equations* (1.1) *and* (4.2)*, respectively. Then, for any* $\beta \in \mathbb{F}_p^\times$*, we have*

$$\sum_{t\in\mathbb{F}_p^\times} \psi(-t\beta) \prod_{i=1}^{r} \mathrm{Kloos}(i,\alpha_i t) = pS_\beta - (-1)^n. \tag{2.4}$$

Proof. By Equations (2.2) and (2.3), we have

$$\begin{aligned}\sum_{t\in\mathbb{F}_p} \psi(-t\beta) \prod_{i=1}^{r} \mathrm{Kloos}(i,\alpha_i t) &= \sum_{t\in\mathbb{F}_p} \psi(-t\beta) \sum_{\beta_1\in\mathbb{F}_p} \psi(t\beta_1) S_{\beta_1} \\ &= \sum_{\beta_1\in\mathbb{F}_p} S_{\beta_1} \sum_{t\in\mathbb{F}_p} \psi(t(\beta_1-\beta)) = pS_\beta.\end{aligned}$$

The identity (2.4) follows by noting that $\mathrm{Kloos}(i,0) = \left(\sum_{x_j\in\mathbb{F}_p^\times} \psi(x_j)\right)^{n_i} = (-1)^{n_i}$. □

2.3. **Combinatorics lemmas.** In the statements of the lemmas and also later, we use the coefficient extractor

$$[z^n]F(z) \overset{\mathrm{def}}{=} f_n \quad \text{for a power series } F(z) = \sum_n f_n z^n.$$

Lemma 8. *Let* $r, n \geq 1$ *be positive integers. Consider a partition* $n = n_1 + \cdots + n_r$*, where each* $n_i \geq 1$*. Define*

$${}_2\binom{n_1,\ldots,n_r}{k} \overset{\mathit{def}}{=} [t^k] \prod_{i=1}^{r} \sum_{j=0}^{n_i} t^{-n_i+2j}.$$

Then, for n *even, we have, for* $-nr \leq 2k \leq nr$, $k \in \mathbb{Z}$,

$$\begin{aligned}{}_2\binom{n_1,\ldots,n_r}{2k} &= \binom{r+\frac{n}{2}-1}{r-1} \\ &+ \sum_{j=1}^{r} (-1)^j \binom{r}{j} \sum_{\substack{n_{i_1}+n_{i_2}+\cdots+n_{i_j} \leq \frac{n}{2}+k-j \\ n_{i_1}\leq n_{i_2}\leq\cdots\leq n_{i_j}}} \binom{r+\frac{n}{2}-\sum_{s=1}^{j} n_{i_s} - j - 1}{r-1}.\end{aligned} \tag{2.5}$$

And, for n *odd, we have, for* $-nr \leq 2k+1 \leq nr$, $k \in \mathbb{Z}$,

$$\begin{aligned}{}_2\binom{n_1,\ldots,n_r}{2k+1} &= \binom{r+\frac{n+1}{2}-1}{r-1} \\ &+ \sum_{j=1}^{r} (-1)^j \binom{r}{j} \sum_{\substack{n_{i_1}+n_{i_2}+\cdots+n_{i_j} \leq \frac{n+1}{2}+k-j \\ n_{i_1}\leq n_{i_2}\leq\cdots\leq n_{i_j}}} \binom{r+\frac{n}{2}-\sum_{s=1}^{j} n_{i_s} - j - 1}{r-1}.\end{aligned} \tag{2.6}$$

Here, and below, we are using the convention for the binomial coefficients that

$$\binom{n}{k} = 0 \quad \textit{if } n < 0 \textit{ or } k > n.$$

We note that this is not the convention in Maple or Mathematica.

Proof. We have, for any positive integers $r, n \geq 1$,

$$\begin{aligned}\prod_{i=1}^{r}\sum_{j=0}^{n_i} t^{-n_i+2j} &= \left(\prod_{i=1}^{r} t^{-n_i}\right)\prod_{i=1}^{r}\left(\sum_{j=0}^{n_i} t^{2j}\right)\\ &= t^{-n}\prod_{i=1}^{r}\frac{1-(t^2)^{n_i+1}}{1-t^2} = t^{-n}\left(\frac{1}{1-t^2}\right)^r\prod_{i=1}^{r}(1-t^{2(n_i+1)})\\ &= t^{-n}\left(\sum_{i=0}^{\infty}\binom{r+i-1}{r-1}t^{2i}\right)\left(1+\sum_{j=1}^{r}(-1)^j\binom{r}{j}\sum_{n_{i_1}\leq n_{i_2}\leq\cdots\leq n_{i_j}} t^{2\sum_{s=1}^{j}(n_{i_s}+1)}\right).\end{aligned}$$

Extracting the coefficients of both sides of the above relation gives Equations (2.5) and (2.6). □

We record the following special cases.

Lemma 9. *If one of r or m (or both) is an even positive integer, then, we have, for $-mr \leq 2k \leq mr$,*

$$ {}_2\binom{\overbrace{m,\ldots,m}^{r \text{ times}}}{2k} = \sum_{i\geq 0}(-1)^i\binom{r}{i}\binom{r+k+\frac{mr}{2}-i(m+1)-1}{r-1}.$$

For r and m both odd positive integers, we have, for $-mr \leq 2k+1 \leq mr$,

$$ {}_2\binom{\overbrace{m,\ldots,m}^{r \text{ times}}}{2k+1} = \sum_{i\geq 0}(-1)^i\binom{r}{i}\binom{r+k+\frac{mr+1}{2}-i(m+1)-1}{r-1}.$$

We apply Lemma 8 to obtain the "multi" Clebsch–Gordan rule.

Lemma 10 (Multi Clebsch–Gordan rule)**.** *Let $r \geq 2$ be an integer. Let $U_{n_i}, (i = 1, \ldots, r)$, be the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ of highest weights n_i. Assume that $n_1 \leq n_2 \leq \cdots \leq n_r$, then*

$$\bigotimes_{i=1}^{r} U_{n_i} \simeq \bigoplus_{k=0}^{K(n_1,\ldots,n_r)} c_k(n_1,\ldots,n_r)U_{n-2k} \tag{2.7}$$

as representations of $\mathfrak{sl}(2, \mathbb{C})$, where $n = n_1 + n_2 + \cdots + n_r$,

$$K(n_1,\ldots,n_r) = \frac{n}{2} - \frac{1}{2}\max\left(\frac{1-(-1)^n}{2}, n_r - n_{r-1} - \cdots - n_1\right),$$

and

$$c_k(n_1,\ldots,n_r) = \sum_{j=0}^{k}(-1)^j\, {}_2\binom{n_1,\ldots,n_r}{n-2k+2j}, \tag{2.8}$$

with ${}_2\binom{n_1,\ldots,n_r}{\cdot}$ given by Lemma 8.

Proof. It suffices to show that the two sides of Equation (2.7) have the same formal character. We have

$$\mathrm{ch}\,(U_{n_i}) = \sum_{j=0}^{n_i} t^{-n_i+2j}.$$

and

$$\operatorname{ch}\left(\bigotimes_{i=1}^{r} U_{n_i}\right) = \prod_{i=1}^{r} \operatorname{ch}\left(U_{n_i}\right) = \prod_{i=1}^{r}\sum_{j=0}^{n_i} t^{-n_i+2j}.$$

Thus, the coefficient $c_0(n,r)$ is the number of times t^n appears in the above product, i.e., $c_0(n,r) = {}_2\binom{n_1,\ldots,n_r}{n}$. Similarly, the coefficient $c_1(n,r)$ is the number of times t^{n-2} appears minus the number of times t^n appears in the above product, i.e., $c_1(n,r) = {}_2\binom{n_1,\ldots,n_r}{n-2} - {}_2\binom{n_1,\ldots,n_r}{n}$. By induction, we obtain Equation (2.8). □

Lemma 11. *Let $P(u) = u^{-2} + 1 + u^2$. Then, we have*

$$\sum_{r=0}^{\infty}[u^0]P(u)^r z^r = \frac{1}{\sqrt{1-2z-3z^2}}. \tag{2.9}$$

As a consequence, we have, as $r \to \infty$,

$$[u^0]P(u)^r = \sqrt{\frac{3}{\pi}}\frac{3^r}{r^{1/2}}\left(1 - \frac{1}{8r} + \frac{1}{128r^2} + O(r^{-3})\right). \tag{2.10}$$

Proof. We have

$$\sum_{r=0}^{\infty}\left(P(u)z\right)^r = \frac{1}{1-zP(u)}.$$

Let $u_1(z)$ and $u_2(z)$ denote the roots of $1 - zP(u) = 0$ such that $\lim_{z\to 0} u_1(z) = \lim_{z\to 0} u_2(z) = 0$. Explicitly,

$$u_1(z) = \frac{\sqrt{-\frac{\sqrt{-3z^2-2z+1}}{z} + \frac{1}{z} - 1}}{\sqrt{2}} \quad \text{and} \quad u_2(z) = -u_1(z).$$

By Cauchy's integral formula for derivatives and residue theorem, we can verify that

$$[u^0]\sum_{r=0}^{\infty}\left(P(u)z\right)^r = \frac{1}{2\pi i}\int_{|u|=\epsilon}\frac{du}{u(1-zP(u))} = z\left(\frac{u_1'(z)}{u_1(z)} + \frac{u_2'(z)}{u_2(z)}\right) = 2z\frac{u_1'(z)}{u_1(z)},$$

which, after simplifications, becomes

$$\frac{1}{\sqrt{-3z^2-2z+1}}. \tag{2.11}$$

This verifies Equation (2.9). To derive Equation (2.10), we use the fact that

$$[z^r](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{\alpha(\alpha-1)}{2r} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24r^2} + O\left(\frac{1}{r^3}\right)\right),$$

and factor the denominator of Equation (2.11) as

$$\sqrt{(-z-1)(\frac{1}{3}-z)} = \frac{1}{\sqrt{3}}\sqrt{(-z-1)(1-3z)}.$$

The closest singularity $1/3$ to the origin dominates, and we obtain Eq. (2.10). □

3. General theory

In this section, we collect known facts about totally wild pure sheaves. We follow [Kat88, Section 7.1] closely.

3.1. **Pure sheaves, totally wild at ∞.** Let X be a smooth projective curve over $\mathbb{F}_p$. Let $U \subsetneq X$ be an open set and put $S = X - U$. Let $j : U \hookrightarrow X$ be the natural inclusion.

Let $\mathcal{F}$ be a lisse sheaf on U, pure of weight w. We have a short exact sequence of sheaves on X:

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to \bigoplus_{s\in S} (j_*\mathcal{F})_s \to 0,$$

which gives the long exact cohomology sequence on X:

$$(3.1)\qquad \begin{array}{ccccc} 0 & \longrightarrow & \mathrm{H}^0(X, j_*\mathcal{F}) & \hookrightarrow & \bigoplus_{s\in S} \mathrm{H}^0\left(X, (j_*\mathcal{F})_s\right) \\ & & & & \downarrow \\ \hookrightarrow \mathrm{H}^1_c(U,\mathcal{F}) & \twoheadrightarrow & \mathrm{H}^1(X, j_*\mathcal{F}) & \longrightarrow & 0. \\ \Vert\,{\scriptstyle \mathrm{def}} & & & & \\ \mathrm{H}^1(X, f_!\mathcal{F}) & & & & \end{array}$$

In the above sequence, the top left $\mathrm{H}^0(X, f_!\mathcal{F}) = 0$ because $\mathcal{F}$ is lisse and X is proper; $\mathrm{H}^i_c(U,\mathcal{F}) = 0 = \mathrm{H}^i(X, j_*\mathcal{F})$ $(\forall i \geq 3)$, and $\mathrm{H}^i\left(X, (j_*\mathcal{F})_s\right) = 0$ $(\forall i \geq 1)$, for dimension reasons. By Deligne's Fundamental Theorem, and by Lemma 2, $\bigoplus_{s\in S} \mathrm{H}^0\left(X, (j_*\mathcal{F})_s\right)$ and $\mathrm{H}^1_c(U,\mathcal{F})$, are mixed of weights $\leq w$ and $\leq 1 + w$, respectively. The remaining groups $\mathrm{H}^0(X, j_*\mathcal{F})$ and $\mathrm{H}^1(X, j_*\mathcal{F})$ are pure of weights w and $1 + w$, respectively, by Theorem A.

If, in addition, that $\mathcal{F}$ is totally wild at some $x \in X - U$, then $\mathrm{H}^2_c(U,\mathcal{F}) = 0$ (by Lemma 3), and $(j_*\mathcal{F})_x = 0$ (see [Kat87, p. 101]), and, thus, the long exact sequence (3.1) becomes the short exact sequence

$$0 \to \bigoplus_{s\in S-\{x\}} (j_*\mathcal{F})_s \hookrightarrow \mathrm{H}^1_c(U,\mathcal{F}) \twoheadrightarrow \mathrm{H}^1(X, j_*\mathcal{F}) \to 0.$$

Hence, in this situation, the parts of $\mathrm{H}^1_c(U,\mathcal{F})$ that are pure of weight $1+w$ and mixed of weight $\leq w$ are isomorphic to $\mathrm{H}^1(X, j_*\mathcal{F})$ and $\bigoplus_{s\in S-\{x\}} \mathrm{H}^0\left(X, (j_*\mathcal{F})_s\right)$, respectively. This reduces the task of counting the lower weights in $\mathrm{H}^1_c(U,\mathcal{F})$ to studying the stalks of the sheaf $j_*\mathcal{F}$ at the remaining missing points $s \in S - \{x\}$. We summarize this discussion in the following

Proposition 1 (Stratification of weights in H^1_c; [Kat88])**.** *Let p be an odd prime, and take $\ell = 2$. Let λ be a 2-adic place of the field $E = \mathbb{Q}(\zeta_p)$, where ζ_p is a root of unity. Let X be a smooth projective curve over $\mathbb{F}_p$. Let $U \subsetneq X$ be an open set and put $S = X - U$. Let $j : U \hookrightarrow X$ be the natural inclusion. Suppose $\mathcal{F}$ is a lisse E_λ-sheaf on U, pure of weight w, and totally wild at some $x \in X - U$. Then, we have the short exact cohomology sequence*

$$(3.2)\qquad 0 \to \bigoplus_{s\in S-\{x\}} (j_*\mathcal{F})_s \hookrightarrow \mathrm{H}^1_c(U,\mathcal{F}) \twoheadrightarrow \mathrm{H}^1(X, j_*\mathcal{F}) \to 0,$$

and in $\mathrm{H}^1_c(U,\mathcal{F})$:

(1) All weights are $\leq 1+w$.

(2) There are canonical isomorphisms of $D_0/I_0 \simeq \mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$*-modules*

$$\mathrm{H}^1(X, j_*\mathcal{F}) \simeq \textit{the part of } \mathrm{H}^1_c(U,\mathcal{F}) \textit{ pure of weight } 1+w$$

and

$$\bigoplus_{s\in S-\{x\}} (j_*\mathcal{F})_s \simeq \textit{the part of } \mathrm{H}^1_c(U,\mathcal{F}) \textit{ of weight } \leq w.$$

(3) The multiplicity of weight $1+w$ *is equal to*

$$h^1_c(U,\mathcal{F}) - \sum_{s\in S-\{x\}} h^0\left(X, (j_*\mathcal{F})_s\right).$$

4. Application to exponential sums

In this section, we apply the method from the previous section to specific sheaves coming from analytic number theory.

Theorem 1 (Main result)**.** *Let* $S_{\beta,\vec{\alpha},p}(r;n_1,\ldots,n_r)$ *be defined as in Equation* (1.1)*. For* p *an odd prime, we have*

$$|S_{\beta,\vec{\alpha},p}(r;n_1,\ldots,n_r)| \leq C(r;n_1,\ldots,n_r)p^{\frac{n-1}{2}} + \sum_{k=1}^{K(r;n_1,\ldots,n_r)} c_k(r;n_1,\ldots,n_r)p^{k-1},$$

where

$$C(r;n_1,\ldots,n_r) = \prod_{i=1}^{r}\left(1+\sum_{j=1}^{n_i}\frac{b_{i,j}}{\mathrm{GCD}(b_{i,j},p)}\right) - \sum_{k=1}^{K(r;n_1,\ldots,n_r)} c_k(r;n_1,\ldots,n_r), \tag{4.1}$$

where

$$K(r;n_1,\ldots,n_r) = \frac{n}{2} - \frac{1}{2}\max\left\{\frac{1-(-1)^n}{2}, n_r - n_{r-1} - \cdots - n_1\right\},$$

and

$$c_k(r;n_1,\ldots,n_r) = \sum_{j=0}^{k}(-1)^j {}_2\binom{n_1,\ldots,n_r}{n-2k+2j},$$

with ${}_2\binom{n_1,\ldots,n_r}{\cdot}$ *given by Lemma 8.*

We make a few remarks before presenting the proof of this theorem.

Remarks 2. *(i) Interpreting cohomologically, the leading constant* $C(r;n_1,\ldots,n_r)$ *is equal to the dimension of* $\mathrm{H}^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)$*, where* $\mathcal{F}$ *is the sheaf on* $\mathbb{G}_m$ *constructed by Equation* (4.3)*.*

(ii) The method of proof could be extended to all primes p*.*

(iii) Katz in [Kat87] deals with the more general exponential sum

$$\sum_{x\in V_{\beta,\vec{\alpha},p^m}} \psi\left(\sum_{\substack{1\leq i\leq r\\ 1\leq j\leq n_i}} x_{i,j}\right)\prod_{\substack{1\leq i\leq r\\ 1\leq j\leq n_i}} \chi_{i,j}(x_{i,j}),$$

where ψ is a non-trivial additive character of $\mathbb{F}_p$ and $\chi_{i,j}$ is any collection of (possibly trivial) multiplicative characters of $\mathbb{F}_p^\times$. In this context, the sum $S_{\beta,\vec{\alpha},p}(r;n_1,\ldots,n_r)$ in Theorem 1 corresponds to the special case $m=1$, $\psi(\cdot)=e_p(\cdot)$ and all the $\chi_{i,j}$ are trivial. Our proof of Theorem 1 extends directly to the case that $m>1$, ψ is non-trivial, and all the $\chi_{i,j}$ are trivial.

(iv) In the complementary case the $\chi_{i,j}$ are not all trivial, the proof of Theorem 1 could be generalized to give an algorithm (depending on the given $\chi_{i,j}$), as opposed to an explicit formula, to compute the number $C(r;n_1,\ldots,n_r)$ of "top weights" in the theorem.

Proof of Theorem 1. We break the proof into three steps.

Step 1. Construction of the corresponding sheaf. In [Kat87], Katz constructed a sheaf on $\mathbb{A}^1$ whose trace is equal to $(-1)^n pS_\beta$, but this sheaf is not necessarily lisse, which then requires a special argument to show that $\mathrm{H}^2_c=0$. We will construct a (slightly different) sheaf on $\mathbb{G}_m$, one that is lisse, so that the wild vanishing Lemma 3 applies directly, provided that the sheaf is also wild at one of the missing points.

Pick a 2-adic place λ of the field $E=\mathbb{Q}(\zeta_p)$, where ζ_p is a root of unity. Denote by $\mathrm{Kloos}(i)$ the lisse E_λ-sheaf on $\mathbb{G}_m$, which is denoted

$$\mathrm{Kl}(\psi;\underbrace{1,1,\ldots,1}_{n_i+1\text{ times}};1,b_{i,1},\ldots,b_{i,n_i})$$

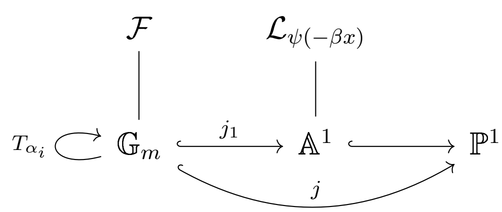
in [Kat88, Theorem 4.1.1]. Its trace function at $t\in\mathbb{G}_m$ is given by

$$\mathrm{tr}\left(\mathrm{Frob}_t \mid \mathrm{Kloos}(i)_{\bar{t}}\right)=(-1)^{n_i}\mathrm{Kloos}(i,t),$$

where

$$\mathrm{Kloos}(i,a)\overset{\text{def}}{=}\sum_{x_{i,1},\ldots,x_{i,n_i}\in\mathbb{F}_p^\times}\psi\left(\sum_{j=1}^{n_i}x_{i,j}+\frac{a}{\prod_{j=1}^{n_i}(x_{i,j})^{b_{i,j}}}\right),\quad (a\in\mathbb{F}_p). \tag{4.2}$$

Let $j:\mathbb{G}_m\hookrightarrow\mathbb{P}^1$ and $j_1:\mathbb{G}_m\hookrightarrow\mathbb{A}^1$ be the natural inclusions, and let T_{α_i} denote the automorphism $x\mapsto\alpha_i x$ of $\mathbb{G}_m$ as depicted below.



The sheaf we are interested in is the following lisse E_λ-sheaf $\mathcal{F}$, pure of weight n, on $\mathbb{G}_m$ defined by

$$\mathcal{F}=\left(\bigotimes_{i=1}^{r}T_{\alpha_i}^{*}Kl(i)\right)\otimes\left({j_1}^{*}\mathcal{L}_{\psi(-\beta x)}\right), \tag{4.3}$$

where $\mathcal{L}_{\psi(-\beta x)}$ is the lisse Artin–Schreier sheaf on $\mathbb{A}^1$ corresponding to the additive character $x\mapsto\psi(-\beta x)$ of $\mathbb{F}_p$. By construction, the trace function at $t\in\mathbb{G}_m$ of $\mathcal{F}$ is given by

$$\mathrm{tr}\left(\mathrm{Frob}_t \mid \mathcal{F}_{\bar{t}}\right)=(-1)^n\psi(-t\beta)\prod_{i=1}^{r}\mathrm{Kloos}(i,\alpha_i t).$$

Summing over $t \in \mathbb{G}_m$ of the above gives, by Lemma 7,

$$\sum_{t\in\mathbb{G}_m} \operatorname{tr}\left(\operatorname{Frob}_t \mid \mathcal{F}_{\bar{t}}\right) = (-1)^n \sum_{t\in\mathbb{G}_m} \psi(-t\beta) \prod_{i=1}^{r} \operatorname{Kloos}(i, \alpha_i t) = (-1)^n p S_\beta - 1.$$

The left side of the above, by the Lefschetz trace formula for $\mathcal{F}$, is equal to

$$\sum_{t\in\mathbb{G}_m} \operatorname{tr}\left(\operatorname{Frob}_t \mid \mathcal{F}_{\bar{t}}\right) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}\left(\operatorname{Frob} \mid \mathrm{H}_c^i\left(\mathbb{G}_m, \mathcal{F}\right)\right).$$

Hence, the original exponential sum is given by

$$S_\beta = \frac{(-1)^n}{p}\left(\sum_{i=0}^{2} (-1)^i \operatorname{tr}\left(\operatorname{Frob} \mid \mathrm{H}_c^i\left(\mathbb{G}_m, \mathcal{F}\right)\right) + 1\right). \tag{4.4}$$

Step 2. Vanishing of H_c^2. By Lemma 3, we have $\mathrm{H}_c^2\left(\mathbb{G}_m, \mathcal{F}\right) = 0$, as $\mathcal{F}$ is lisse on $\mathbb{G}_m$ and totally wild at $\infty \in \mathbb{P}^1 - \mathbb{G}_m$. We also have $\mathrm{H}_c^0\left(\mathbb{G}_m, \mathcal{F}\right) = 0$, since $\mathcal{F}$ is lisse and $\mathbb{P}^1$ is proper. Thus, by Equation (4.4), we obtain

$$S_\beta = \frac{(-1)^n}{p}\left(-\operatorname{tr}\left(\operatorname{Frob} \mid \mathrm{H}_c^1\left(\mathbb{G}_m, \mathcal{F}\right)\right) + 1\right). \tag{4.5}$$

We record the next lemma for latter use.

Lemma 12. *For $\beta \neq 0$, the cohomology group $\mathrm{H}_c^1\left(\mathbb{G}_m, \mathcal{F}\right)$ has dimension*

$$h_c^1 = \prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} \frac{b_{i,j}}{\operatorname{GCD}(b_{i,j}, p)}\right). \tag{4.6}$$

Proof. For $\beta \neq 0$, H_c^1 is the only non-vanishing cohomology group, so

$$\chi_c\left(\mathbb{G}_m, \mathcal{F}\right) = -h_c^1.$$

Equation (4.6) follows directly from the above equation. ☐

Step 3. Stratification of weights in H_c^1. We now apply Proposition 1 with $U = \mathbb{G}_m$, $X = \mathbb{P}^1$, (hence, $S = \{0, \infty\}$), and with $\mathcal{F}$ the lisse E_λ-sheaf on U, pure of weight n, totally wild at ∞, given by Equation (4.3). Then, we have the short exact sequence

$$0 \to \left(j_*\mathcal{F}\right)_0 \hookrightarrow \mathrm{H}_c^1(U, \mathcal{F}) \twoheadrightarrow \mathrm{H}^1(X, j_*\mathcal{F}) \to 0, \tag{4.7}$$

and in $\mathrm{H}_c^1(U, \mathcal{F})$:

(1) All weights are $\leq 1 + n$.
(2) There is a canonical isomorphism of $\operatorname{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$-modules

$$\mathrm{H}^1(X, j_*\mathcal{F}) \simeq \text{the part of } \mathrm{H}_c^1(U, \mathcal{F}) \text{ pure of weight } 1 + n \tag{4.8}$$

and

$$\left(j_*\mathcal{F}\right)_0 \simeq \text{the part of } \mathrm{H}_c^1(U, \mathcal{F}) \text{ of weight} \leq n. \tag{4.9}$$

(3) The multiplicity of the top weight $1 + n$ is equal to $h_c^1(\mathcal{F})$ minus the dimension of the stalk $\left(j_*\mathcal{F}\right)_0$.

Thus, by Equations (4.8) and (4.9), Eq. (4.5) becomes

$$S_\beta = \frac{(-1)^n}{p}\left(-\mathrm{tr}\left(\mathrm{Frob}\mid \mathrm{H}^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)\right) - \mathrm{tr}\left(\mathrm{Frob}\mid (j_*\mathcal{F})_0\right) + 1\right). \tag{4.10}$$

By Theorem A, $\mathrm{H}^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)$ is pure of weight $1+n$ and we have

$$\mathrm{tr}\left(\mathrm{Frob}\mid \mathrm{H}^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)\right) = \sum_{i=1}^{h^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)} \gamma_i \tag{4.11}$$

with

$$|\gamma_i| = p^{(n+1)/2}, \text{ for each } i = 1, 2, \ldots, h^1\left(\mathbb{P}^1, j_*\mathcal{F}\right).$$

Next, by definition of $\mathcal{F}$, we have

$$(j_*\mathcal{F})_0 \simeq \left(\left(\bigotimes_{i=1}^{r} j_*T^*_{\alpha_i} Kl(i)\right) \otimes j_*j_1^*\mathcal{L}_{\psi(-\beta x)}\right)_0.$$

As I_0-representations, the sheaves $j_*T^*_{\alpha_i}Kl(i)$ and $j_*j_1^*\mathcal{L}_{\psi(-\beta x)}$ "are" a single unipotent block $U_{n_i}, (\dim U_{n_i} = 1 + n_i)$ and an identity matrix of dimension 1, respectively. Thus,

$$(j_*\mathcal{F})_0 \simeq \bigotimes_{i=1}^{r} U_{n_i}.$$

By Equation (2.7), the above tensor product decomposes to

$$\bigotimes_{i=1}^{r} U_{n_i} \simeq \bigoplus_{k=0}^{K(n_1,\ldots,n_r)} c_k(n_1,\ldots,n_r) U_{n-2k},$$

where $c_k(n_1,\ldots,n_r)$ is given explicitly by Equation (2.8). Each U_{n-2k} is pure of (real) weight $2k$ and adds one dimension to the stalk $(j_*\mathcal{F})_0$. Each coefficient $c_k(n_1,\ldots,n_r)$ thus gives the number of weights $= 2k$ in $(j_*\mathcal{F})_0$, $(0 \le k \le K(n_1,\ldots,n_r))$. Thus,

$$\mathrm{tr}\left(\mathrm{Frob}\mid (j_*\mathcal{F})_0\right) = \sum_{k=0}^{K(n_1,\ldots,n_r)} c_k(n_1,\ldots,n_r)p^k.$$

Hence, by the above and Equation (4.11), Eq. (4.10) becomes

$$S_\beta = \frac{(-1)^n}{p}\left(-\sum_{i=1}^{h^1\left(\mathbb{P}^1, j_*\mathcal{F}\right)} \gamma_i - \sum_{k=0}^{K(n_1,\ldots,n_r)} c_k(n_1,\ldots,n_r)p^k + 1\right).$$

Noting that, by the short exact sequence (4.7),

$$h^1\left(\mathbb{P}^1, j_*\mathcal{F}\right) = h^1_c\left(\mathbb{G}_m, \mathcal{F}\right) - \sum_{k=0}^{K(n_1,\ldots,n_r)} c_k(n_1,\ldots,n_r), \tag{4.12}$$

where h^1_c given by Equation 4.6, this completes the proof of Theorem 1. $\square$

Remarks 3. *(i) We were lead to compute $h^1(\mathbb{P}^1, j_*\mathcal{F})$, which is equal to $C(r; n_1, \ldots, n_r)$, via the right side of Equation (4.12), as we were not able to compute $h^1(\mathbb{P}^1, j_*\mathcal{F})$ directly.*

(ii) The constant $C(r; n_1, \ldots, n_r)$ depends on $b_{j,k}$ via $h^1_c(\mathbb{G}_m, \mathcal{F})$ given by Equation 4.6, but is independent of p, β, and $\vec{\alpha}$.

(iii) If we define the angles $\theta_j \in [0, 2\pi)$ by $\gamma_j = p^{(n+1)/2} e^{i\theta_j}$, where γ_i given by Equation 4.11, then the dependence on p, β, and $\vec{\alpha}$ appears in these angles $\theta_j = \theta_j(p, \beta, \vec{\alpha})$. Our proof of Theorem 1, unfortunately, gives no information about the distribution of the angles $\theta_j(p, \beta, \vec{\alpha})$ as p (or β, or $\vec{\alpha}$) varies. We hope to investigate this problem in a future article.

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