On the class number of quadratic number fields

by
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Through an ingenious combination of the two approaches by Hecke and Deuring, Heilbronn has succeeded in proving the long-supposed theorem that the class number $h(d)$ of the imaginary quadratic number field of discriminant d becomes infinite with $|d|$. It is natural to ask for a more accurate lower bound for $h(d)$. The following is the expected asymptotic formula

$$\log h(d) \sim \log \sqrt{|d|} \tag{1}$$

to be proven. Dirichlet proved that

$$\pi |d|^{-\frac{1}{2}} h(d) = L_d(1) \qquad (d < -4),$$

where

$$L_d(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s},$$

thus, (1) is equivalent to the statement

$$\log L_d(1) = o(\log |d|). \tag{2}$$

One will assume that (2) also holds for positive discriminants d; and this will also be proved. Indeed, if ε_d means the fundamental unit, then according to Dirichlet,

$$2d^{-\frac{1}{2}} h(d) \log \varepsilon_d = L_d(1) \qquad (d > 0),$$

and, consequently, the relationship

$$\log(h(d) \log \varepsilon_d) \sim \log \sqrt{d}$$

[1] This is a translation of Siegel, C. L., Über die Classenzahl quadratischer Zahlkörper." Acta Arith. **1** (1935) pp. 83-86. It is available at https://mast.queensu.ca/∼dnguyen/translations

is the analogue of (1) for real quadratic fields. One shall further ask whether (2) could be generalized to arbitrary algebraic number fields. Since $L_d(1)$ is equal to the residue of the zeta function of the quadratic number field of discriminant d, then, on the basis of Dedekind's class number formula, for any algebraic number field of fixed degree with discriminant d, class number h, and regulator R, one expects that

$$\log(hR) \sim \log\sqrt{|d|} \tag{3}$$

holds true. Using class field theory, the method presented here could be used to reduce the task of proving (3) to the set of all solvable fields with respect to an arbitrary fixed algebraic number field. The general case however could well be inaccessible as long as the decomposition laws of non-solvable fields remain unknown.

For the proof of (2) it seems appropriate to modify Heilbronn's idea in such a way that the asymptotic expansion of the series $\sum_{n=-\infty}^{\infty}(an^2+bn+c)^{-s}$ originating from Deuring is no longer needed for $4ac-b^2\to\infty$ and instead the Hecke estimate is transferred to that of the zeta function of a biquadratic field composed of two quadratic number fields.

Let $\mathfrak{K}$ be an algebraic number field of degree n with discriminant d; of its conjugates, let r_1 be the number of real and r_2 of complex conjugate pairs. For brevity, let $r_1+r_2=q$. If $x_1,\dots,x_q$ are positive variables and $x_{r_2+\ell}=x_\ell$ $(\ell=r_1+1,\dots,r_1+r_2)$, set $\prod_{k=1}^n x_k = Nx$ and $\sum_{k=1}^n x_k=\sigma(x)$. Let $\zeta(s,\mathfrak{K})$ be the zeta function of $\mathfrak{K}$ and let χ be its residue at $s=1$, further set

$$(2\pi)^{-r_2}|d|^{\frac{1}{2}}\chi=\lambda. \tag{4}$$

Lemma 1:

In the entire complex plane,

$$\begin{aligned} 2^{-r_2 s}\pi^{-\frac{n}{2}s}|d|^{\frac{s}{2}}\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)\zeta(s,\mathfrak{K}) &= \frac{\lambda}{s(s-1)}+ \\ &+\sum_{\mathfrak{a}}\int_{Nx\geq 1}\cdots\int\left(Nx^{\frac{s}{2}}+Nx^{\frac{1-s}{2}}\right)e^{-\pi(N\mathfrak{a})^{\frac{2}{n}}|d|^{-\frac{1}{n}}\sigma(x)}\frac{dx_1}{x_1}\cdots\frac{dx_q}{x_q}, \end{aligned} \tag{5}$$

where $\mathfrak{a}$ runs through all integral ideals of $\mathfrak{K}$.

The proof follows directly from the integral representation of $\zeta(s,\mathfrak{K})$, which Hecke used to prove the functional equation. □

Lemma 2:
Let $0 < s < 1$ and $\zeta(s, \mathfrak{K}) \leq 0$. Then

$$\chi > s(1-s)2^{-n}e^{-2n\pi}|d|^{\frac{s-1}{2}}. \tag{6}$$

Proof: For real s, all terms of the infinite series on the right side of (5) are positive. The term with $\mathfrak{a} = 0$ is reduced if the q-fold integral is only over the q-dimensional cube $|d|^{\frac{1}{n}} \leq x_\ell \leq 2|d|^{\frac{1}{n}}$ ($\ell = 1, \ldots, q$); but in this the integrand is at least $|d|^{\frac{s}{2}}e^{-2n\pi}2^{-q}|d|^{-\frac{q}{n}}$. Hence the infinite series is at least $|d|^{\frac{s}{2}}e^{-2n\pi}2^{-n}$. If $0 < s < 1$ and $\zeta(s, \mathfrak{K}) \leq 0$, then by Lemma 1,

$$\frac{\lambda}{s(s-1)} + |d|^{\frac{s}{2}}e^{-2n\pi}2^{-n} < 0.$$

So, by (4), the assertion (6) is proven. □

Let d be the discriminant of a quadratic number field $\mathfrak{K}_1$. If D is the discriminant of a quadratic number field different from $\mathfrak{K}_1$ and t is the discriminant of the field generated by $\sqrt{dD}$, then $t^{-1}dD$ is an integer, i.e.,

$$|t| \leqq |dD|. \tag{7}$$

Let $\mathfrak{K}_2$ be the biquadratic number field generated by $\sqrt{d}$ and $\sqrt{D}$. As a special case of well-known theorems of class field theory, we have

Lemma 3: The discriminant of $\mathfrak{K}_2$ is equal to dDt and

$$\zeta(s, \mathfrak{K}_1) = \zeta(s)L_d(s),$$
$$\zeta(s, \mathfrak{K}_2) = \zeta(s)L_d(s)L_D(s)L_t(s).$$

Furthermore, one obtains by partial summations the following

Lemma 4:
We have

$$L_d(1) < 3\log|d|.$$

Using the last three lemmas, the proof of (2) can now be carried out as follows. If (2) were false, then there would be a positive $\epsilon < 1$ and $|d|$ of any size, such that $L_d(1) > |d|^\epsilon$ or $L_d(1) < |d|^{-\epsilon}$. Suppose

$$3\log|d| < |d|^\epsilon. \tag{8}$$

Then, by Lemma 4, the case must be

$$L_d(1) < |d|^{-\epsilon}. \tag{9}$$

Also,

$$|d|^{\frac{\epsilon}{2}} > \frac{4e^{4\pi}}{\epsilon(1-\epsilon)}, \tag{10}$$

so (9) yields the inequality

$$L_d(1) < (1-\epsilon)\epsilon 2^{-2} e^{-4\pi} |d|^{-\frac{\epsilon}{2}}.$$

According to Lemmas 2 and 3 then $\zeta(1-\epsilon, \mathfrak{R}_1) > 0$. Furthermore, fix a d that satisfies the inequalities (8) and (10).

The function $\zeta(s, \mathfrak{R}_1)$ becomes negatively infinite when s tends towards 1. So it has a root σ in the interval $1-\epsilon < \sigma < 1$. According to Lemma 3, $\zeta(\sigma, \mathfrak{R}_2) = 0$. If one applies Lemma 2 with $\mathfrak{R} = \mathfrak{R}_2$ and uses Lemma 3, one obtains the equation

$$L_d(1) L_D(1) L_t(1) > \sigma(1-\sigma) 2^{-4} e^{-8\pi} |dDt|^{\frac{\sigma-1}{2}}.$$

According to (7) and Lemma 4, it therefore holds that

$$L_D(1) > \frac{\sigma(1-\sigma)}{3^2 \cdot 2^4 \cdot e^{8\pi} \log|d| \log|dD|} |dD|^{\sigma-1},$$

and, consequently, because $\sigma - 1 > -\epsilon$ for all sufficiently large $|D|$, the estimate

$$L_D(1) > |D|^{-\epsilon}$$

is valid. So (9) can only hold for finitely many d, contradicting the assumption that (2) is false. □

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