Performance Analysis of MAP Decoded Space-Time Orthogonal Block Codes for Non-Uniform Sources*

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Abstract — **We derive a closed-form expression for the exact pairwise error probability (PEP) of a non-uniform memoryless binary source transmitted over a Rayleigh fading channel using space-time orthogonal block codes and maximum *a posteriori* (MAP) detection. The expression is easy to evaluate and holds for any signaling scheme. We then use this result to minimize the bit error rate of the binary antipodal signaling scheme. Numerical results for the case of binary antipodal signaling (BPSK and optimal) verify the accuracy of our formula and quantify substantial gains of MAP decoding over maximum likelihood (ML) decoding for sources with strong non-uniformity.**

I. Introduction

The original papers published on space-time orthogonal block (STOB) codes [11] adopt the Chernoff upper bound to estimate the pairwise error probability of codewords and derive code design criteria. Although the Chernoff bound results in successful code constructions, it is quite loose even at high values of channel signal-to-noise ratio (CSNR). It is common practice in the literature to use the union bound to approximate the system symbol error rate (SER) or bit error rate (BER). However, the union bound is intrinsically loose, particularly at low CSNRs. Therefore, using the Chernoff bound together with the union bound results in poor approximations to system performance.

For maximum likelihood (ML) decoding, the main challenge in finding the PEP of the symbols is to find the expected value of $Q(\sqrt{X})$ where $Q(\cdot)$ is the Gaussian Q-function and X is a non-negative random variable. In [12], an expression for the exact PEP of space-time trellis codes is found and used to derive an upper bound on the BER. Another way to numerically compute the PEP is described in [10], which is easier to compute in certain cases. In [4], simple formulas in closed-form for the exact PEP of space-time (trellis and block orthogonal) coded equally likely symbols are established for the case of ML decoded slow Rayleigh fading channels and very tight upper and lower bounds on system SER and BER are derived.

In this work, we consider a non-uniform memoryless binary source and provide the exact PEP for STOB coded transmission of symbols for any two dimensional constellation. We also examine how the exploitation of the source non-uniformity at the transmitter and/or the receiver can improve the performance of STOB coded systems. In particular, the effects of MAP decoding and constellation design for binary signaling are addressed. As will be seen in the sequel, MAP detection requires finding the expected value of $Q(\sqrt{X}+\frac{\lambda}{\sqrt{X}})$, where λ is a real number. This is more involved than the ML decoding case. However, we demonstrate that, when the source is strongly non-uniform, there can be a large gain in performing MAP decoding as compared with ML decoding. This is consistent with previous joint source-channel coding work regarding the transmission of non-uniform sources in single antenna systems (e.g., [1, 2, 5, 9, 13]). To the best of our knowledge, there is no work in the literature on performance analysis or simulation of space-time codes under MAP decoding.

II. System Model

The multi-antenna communication system considered here employs L_T transmit and L_R receive antennas. The input to the system is an independent and identically distributed (i.i.d.) bit-stream $\{b_i\}$ which can have non-uniform distribution. The complex baseband constellation points are denoted by $\{c_k\}_{k=1}^{2^q}$ where q is a positive integer. We assume that the signal constellation has an average energy of one, i.e., $\sum_{k=1}^{2^q} p(c_k).|c_k|^2 = 1$, where $p(c_i) = \prod_{j=1}^{q} p(b_j^i)$ and $b_1^i...b_q^i$ is the bit-string corresponding to symbol c_i. In order to have an average transmit power of E_s, the modulators weight the symbols by[1] $\sqrt{\frac{E_s}{L_T}}$. The channel is assumed to be Rayleigh flat fading, so that the complex path gain from transmit antenna i to receive antenna j, denoted by H_{ji}, has a zero-mean unit-variance complex Gaussian distribution, denoted by $\mathcal{CN}(0,1)$, with i.i.d. real and imaginary parts. We assume that the receiver, but not the transmitter, has perfect knowledge of the path gains. Moreover, we assume that the channel is quasi-static, meaning that the path gains remain constant during a codeword transmission, but vary in an i.i.d. fashion from one codeword interval to the other. The additive noise at receiver j at symbol interval t, N_t^j, is assumed to be $\mathcal{CN}(0,\sigma_n^2)$ distributed with i.i.d. real and imaginary parts. For a CSNR of $\gamma_s = \frac{E_s}{\sigma_n^2}$ at each receive antenna and at time t, the signal at receive antenna j can be written as $R_t^j = \sqrt{\frac{E_s}{L_T}} \sum_{i=1}^{L_T} H_{ji} s_t^i + N_t^j$, where $\sqrt{\frac{E_s}{L_T}} s_t^i$ is the signal sent from antenna i, or in matrix form,

$$\boldsymbol{r}_t = \sqrt{\frac{E_s}{L_T}} \boldsymbol{H} \mathbf{s}_t + \boldsymbol{n}_t, \tag{1}$$

[1] Due to the structure of STOB codes, the constellation point representing each $q-$tuple of information bits is used by each antenna an equal number of times; thus the average signal energy for each antenna is the same (and equals $\frac{E_s}{L_T}$).

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where $\boldsymbol{r}_t = (R_t^1, ..., R_t^{L_R})^T$, $\mathbf{s}_t = (s_t^1, ..., s_t^{L_T})^T$, $\boldsymbol{n}_t = (N_t^1, ..., N_t^{L_R})^T$, and ^{T} denotes transposition. $\boldsymbol{H}$ is the $L_R \times L_T$ path gains matrix with elements $\{H_{ji}\}$.

III. The MAP Decoding Rule

In the case of STOB codes with a codeword length of T_w symbol intervals, (1) can be written as [7]

$$\bar{\boldsymbol{r}}^j = \sqrt{\frac{E_s}{L_T}} \bar{\boldsymbol{H}}^j \mathbf{c} + \bar{\boldsymbol{n}}^j \qquad j = 1, \cdots, L_R, \quad (2)$$

where $\bar{R}_t^j = R_t^j$ and $\bar{N}_t^j = N_t^j$ for $1 \leq t \leq \frac{T_w}{2}$, $\bar{R}_t^j = R_t^{j*}$ and $\bar{N}_t^j = N_t^{j*}$ for $\frac{T_w}{2} < t \leq T_w$, $\mathbf{c}$ indicates the $\tau \times 1$ vector of transmitted symbols (τ is a function of L_T and the STOB code), * indicates complex conjugation, and $\bar{\boldsymbol{H}}^j$ is derived from the j^{th} row of $\boldsymbol{H}$ via negation and/or complex conjugation of some of its entries. It is clear that $\bar{N}_t^j$ are i.i.d.$\mathcal{CN}(0, \sigma_n^2)$.

The matrix $\bar{\boldsymbol{H}}^j$ has orthogonal columns, i.e., $\bar{\boldsymbol{H}}^{j\dagger} \bar{\boldsymbol{H}}^j = gY_j \mathbf{I}_\tau$, where † denotes the complex conjugate transpose operation, $\mathbf{I}_n$ is the $n \times n$ identity matrix, $Y_j = \sum_i |H_{ji}|^2$, and g is the inverse of the code rate. Therefore, (2) can be multiplied from the left by $\bar{\boldsymbol{H}}^{j\dagger}$ to yield

$$\tilde{\boldsymbol{r}}^j \triangleq \bar{\boldsymbol{H}}^{j\dagger} \bar{\boldsymbol{r}}^j = g \sqrt{\frac{E_s}{L_T}} Y_j \mathbf{c} + \tilde{\boldsymbol{n}}^j,$$

where $\tilde{\boldsymbol{n}}^j \triangleq \bar{\boldsymbol{H}}^{j\dagger} \bar{\boldsymbol{n}}^j$. Note that each entry of $\tilde{\boldsymbol{r}}^j$ is associated with only *one* symbol. It is not hard to verify that the distribution of the noise sample $\tilde{N}_k^j$ at an arbitrary symbol interval k and an arbitrary receive antenna j is given by

$$\tilde{N}_k^j \sim \text{i.i.d. } \mathcal{CN}(0, gY_j \sigma_n^2). \quad (3)$$

This shows that the noise vector $\tilde{\boldsymbol{n}}^j$ is composed of i.i.d. random variables, hence symbol i can be detected by only considering the i^{th} entry of the vectors $\tilde{\boldsymbol{r}}^j$, $1 \leq j \leq L_R$. Since, given $\boldsymbol{H}$, $\tilde{\boldsymbol{R}}$ is an invertible function of $\boldsymbol{R}$, MAP decoding can be based on $\tilde{\boldsymbol{R}}$ instead of $\boldsymbol{R}$ in the following way

$$\begin{aligned}
c_i &= \arg\max_c P(c | \{\tilde{R}_i^l\}_{l=1}^{L_R}, \boldsymbol{H}) \\
&= \arg\max_c f(\{\tilde{R}_i^l\}_{l=1}^{L_R} | c, \boldsymbol{H}) \cdot p(c) \\
&= \arg\max_c \prod_{l=1}^{L_R} f_{\tilde{N}_i^l}(\{\tilde{R}_i^l - g' Y_l c\}_{l=1}^{L_R}) \cdot p(c) \qquad (4) \\
&= \arg\max_c \left\{ \ln(p(c)) - \sum_{l=1}^{L_R} \frac{|\tilde{R}_i^l - g' Y_l c|^2}{g Y_l \sigma_n^2} \right\}, \quad (5)
\end{aligned}$$

where $g' = g\sqrt{\frac{E_s}{L_T}}$, and (4) and (5) hold because $\tilde{N}_k^j$ are i.i.d. and Gaussian, respectively, as shown in (3).

IV. Exact Pairwise Error Probability with MAP Detection

A. The Conditional PEP

Without loss of generality, we consider MAP decoding for the k^{th} symbol period. The error probabilities may be determined using the MAP detection metric given in (5). The receiver should evaluate this metric for all symbols given that c_i is transmitted (hence $\tilde{R}_k^l = g' Y_l c_i + \tilde{N}_k^l$) and decide in favor of the one which yields a larger result. Denoting the probability that "c_j has a larger metric than c_i when c_i is sent" by $P(\hat{c} = c_j | c = c_i)$, we want the probability of the event that the expression between the brackets in (5) is larger for c_j. This event is equivalent to

$$\sum_{l=1}^{L_R} \frac{\sqrt{2} \langle c_j - c_i, \tilde{N}_k^l \rangle}{d_{ji} \sigma_n} \geq \frac{1}{d_{ji}} \sqrt{\frac{L_T}{2\gamma_s}} \ln \frac{p(c_i)}{p(c_j)} + \frac{g' d_{ji}}{\sigma_n \sqrt{2}} \sum_{l=1}^{L_R} Y_l, \quad (6)$$

where $d_{ji} = |c_j - c_i|$, $\langle x, y \rangle = \Re\{x\}\Re\{y\} + \Im\{x\}\Im\{y\}$, and $\Re\{\cdot\}$ and $\Im\{\cdot\}$ indicate real and imaginary parts, respectively. From (3), it follows that $\frac{\sqrt{2}\langle c_j - c_i, \tilde{N}_k^l \rangle}{|c_j - c_i| \sigma_n}$ are i.i.d.$\mathcal{N}(0, gY_l)$. Hence the distribution of the sum on the left hand side of (6) is $\mathcal{N}(0, g\sum_{l=1}^{L_R} Y_l)$, and the probability of the event in (6), which is the PEP conditioned on the path gains, is given by

$$P(\hat{c} = c_j | c = c_i, \boldsymbol{H}) = Q\left(d_{ji} \sqrt{\frac{g\gamma_s}{2L_T}} \sqrt{Y} + \frac{1}{d_{ji}} \sqrt{\frac{L_T}{2g\gamma_s}} \ln \frac{p(c_i)}{p(c_j)} \frac{1}{\sqrt{Y}} \right), (7)$$

where $Y = \sum_{l=1}^{L_R} Y_l = \sum_{k=1}^{L_T} \sum_{l=1}^{L_R} |H_{lk}|^2$ is the sum of the squared magnitudes of all path gains.

B. The Average (Unconditional) PEP

To find the unconditional PEP, one should average (7) with respect to Y. Using the moment generating function of normal random variables, one can verify that Y is a chi-squared random variable with $2n$ degrees of freedom, with a probability density function (pdf) given by

$$f_Y(y) = \frac{1}{(n-1)!} y^{n-1} e^{-y}, \qquad y > 0.$$

The average of (7) can then be written as

$$P(\hat{c} = c_j | c = c_i) = \frac{1}{(n-1)! \delta_{ij}^{2n}} \int_0^\infty y^{n-1} e^{-y/\delta_{ij}^2} Q\left(\sqrt{y} + \frac{\lambda_{ij}}{\sqrt{y}}\right) dy, \quad (8)$$

where $\delta_{ij} = \sqrt{\frac{g\gamma_s}{2L_T}} |c_i - c_j|$ and $\lambda_{ij} = \frac{1}{2} \ln \frac{p(c_i)}{p(c_j)}$. We note that the integral in (8) is the Laplace transform of $y^{n-1} Q\left(\sqrt{y} + \frac{\lambda_{ij}}{\sqrt{y}}\right)$ evaluated at $s = \delta_{ij}^{-2}$. We know that if $f(t)$ and $F(s)$ are Laplace transform pairs ($F(s) = \mathcal{L}\{f(t)\}$), so are $t^n f(t)$ and $(-1)^n \frac{d^n}{ds^n} F(s)$. Therefore, we need to find the $(n-1)^{\text{st}}$ derivative of

$$\mathcal{L}\left\{ Q\left(\sqrt{y} + \frac{\lambda_{ij}}{\sqrt{y}}\right) \right\} = \frac{1 - \operatorname{sgn}(\lambda_{ij})}{2s} - \frac{1}{2} \left(\frac{1}{s\sqrt{2s+1}} - \frac{\operatorname{sgn}(\lambda_{ij})}{s} \right) e^{-(\lambda_{ij} + |\lambda_{ij}| \sqrt{2s+1})}, \ (9)$$

where $\operatorname{sgn}(x) = \frac{|x|}{x}$ if $x \neq 0$ and 0 otherwise. We find the $(n-1)^{\text{st}}$ derivative of (9) using induction and some algebra. The result is equation (10) which gives the exact PEP of MAP decoded space-time orthogonal block codes $\left(\alpha_{n,k}^l = \sum_{p=0}^{l-1} \binom{l}{p} (-1)^{l+p} \prod_{q=0}^{n-k-2} (l - p - 2q)\right)$.

$$P(\hat{c}=c_j|c=c_i)=\frac{1-\mathrm{sgn}(\lambda_{ij})}{2}-\frac{1}{2}e^{-\left(\lambda_{ij}+|\lambda_{ij}|\sqrt{2\delta_{ij}^{-2}+1}\right)}$$
$$\times\sum_{k=0}^{n-1}\left\{\frac{(-1)^{n+k-1}}{(\delta_{ij}^2+2)^{n-k-1}}\left(\mathrm{sgn}(\lambda_{ij})+\frac{\delta_{ij}}{\sqrt{\delta_{ij}^2+2}}\sum_{m=0}^{k}\binom{2m}{m}\frac{1}{(2\delta_{ij}^2+4)^m}\right)\sum_{l=1}^{n-k-1}\alpha_{n,k}^l\frac{|\lambda_{ij}|^l(\delta_{ij}^2+2)^{l/2}}{l!\delta_{ij}^l}\right\} \quad (10)$$

C. The PEP of ML Decoded STOB Codes

For uniform sources, we have $p(c_i)=p(c_j),\forall i,j$, and MAP decoding reduces to ML decoding. In this case, $\lambda_{ij}=\frac{1}{2}\ln\frac{p(c_i)}{p(c_j)}=0$. Hence, the first sum in (10) is non-zero only for $k=n-1$ (we assume that $\sum_{i=L}^{U}z_i=1$ if $L>U$), and we have

$$P(\hat{c}=c_j|c=c_i)=\frac{1}{2}\left(1-\frac{\delta_{ij}}{\sqrt{2+\delta_{ij}^2}}\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{(2\delta_{ij}^2+4)^k}\right),$$

which agrees with the result we derived in [4].

V. The Optimum Binary Antipodal Signaling

In this section we consider binary antipodal signaling and optimize it in the sense of minimizing the BER given by

$$\mathrm{BER}=P(\hat{c}=c_2|c=c_1)\cdot p(c_1)+P(\hat{c}=c_1|c=c_2)\cdot p(c_2). \quad (11)$$

Normally, one should use the averaged PEP in (11) with $c_1=a$ and $c_2=-b$, differentiate the result, and find the optimal a and b. However, this can be a tedious job considering the PEP given in (10). Therefore, we use the PEPs at the *receiver* side, i.e., given $\boldsymbol{H}$, to find the solution in an easier way. The optimal constellation derived in this way will not depend on $\boldsymbol{H}$, justifying our approach.

Let us assume that $p(c_1)=p$, and the bits 0 and 1 are mapped to $c_2=-b$ and $c_1=a$, respectively. Letting $\beta=\sqrt{\frac{2g\gamma_s}{L_T}}$, $\sqrt{A}=\beta\frac{(a+b)}{2}\sqrt{y}$, and $B=\frac{1}{2}\ln\frac{1-p}{p}$, with Y as defined below (7), we can write the BER conditioned on $\boldsymbol{H}$ as

$$\mathrm{BER}_Y=pQ\left(\sqrt{A}-\frac{B}{\sqrt{A}}\right)+(1-p)Q\left(\sqrt{A}+\frac{B}{\sqrt{A}}\right). \quad (12)$$

It is easy to verify that the BER is a strictly decreasing function of A (regardless of B). Hence, given E_s and p, in order to minimize the BER, one has to maximize A. Note that A is a scaled distance between the constellation points, therefore, signaling schemes with the same distance between their signals have identical performance. It is clear that the constellation with constant average signal energy $\frac{E_s}{L_T}$ which maximizes A is the zero-mean constellation, because a constellation with a non-zero mean can simply be shifted to reduce its energy without performance loss.

From the zero-mean condition, we have $b=\frac{p}{1-p}a$, and the average energy condition requires that

$$pa^2+(1-p)b^2=\frac{E_s}{L_T}\cdot$$

The above two equalities result in

$$(-b,a)=\sqrt{\frac{E_s}{L_T}}\left(-\sqrt{\frac{p}{1-p}},\sqrt{\frac{1-p}{p}}\right),$$

which is therefore the optimal binary antipodal constellation.

The above constellation is identical to the antipodal signaling result in [6] for the case of the AWGN channel.

VI. Simulation Results

It suffices to study the BER of binary antipodal signaling to show the exactness of our PEP formulas. We simulate the transmission of an i.i.d. bit sequence over the MIMO channel. The length of the bit-sequence is $\min(\frac{100}{\mathrm{BER}},10^5)$ bits.

We consider a system with two transmit and one receive antennas using Alamouti's space-time code [3] in Figure 1. We remark that the analysis and simulation curves coincide everywhere. This simulation also indicates how MAP decoding can improve performance. The MAP decoding gain over ML decoding is 0.63, 1.57, and 5.98 dB for $p=0.8,0.9$, and 0.99, respectively, at BER=10^{-3}.

Figure 2 shows the analysis and simulation curves for a system with a uniform source ($p=0.5$), three transmit and various receive antennas, which employs the code $\mathcal{G}^3$ in [11]. As expected, using more receive antennas improves system performance. For example, at BER = 10^{-4} there is a gain of 6.9, 9.5, and 11.3 dB in using 2, 3, and 4 receive antennas instead of only one.

Figure 3 presents the results of Section V. It shows that a large gain can be obtained through optimization of the constellation according to the prior probabilities. At a BER of 10^{-3}, the gain of using the modified constellation and MAP detection is 2.7, 6.1, and 20.1 dB, for $p=0.8,0.9$, and 0.99, respectively, over an ML decoded system with BPSK modulation.

In Figure 4 we compare four systems, i.e., two systems with BPSK signaling and ML or MAP decoding, and two systems with optimum signaling and ML or MAP decoding. These systems are indicated by ML symmetric, MAP symmetric, ML optimum, and MAP optimum, respectively. We note that if the CSNR is high enough, optimum signaling and ML detection outperforms BPSK and MAP detection. This is because when receiver noise is strong (low γ_s), the second term in the argument of the Q function in (7) has a dominant effect; hence MAP decoding and BPSK signaling is more effective than ML decoding and optimum signaling. In less noisy channel conditions (high γ_s), the first term in the argument of the Q function becomes dominant and hence ML decoding with optimum signaling outperforms MAP decoding with symmetric signals. As previously mentioned, MAP decoding with optimum signaling is always better than the other systems.

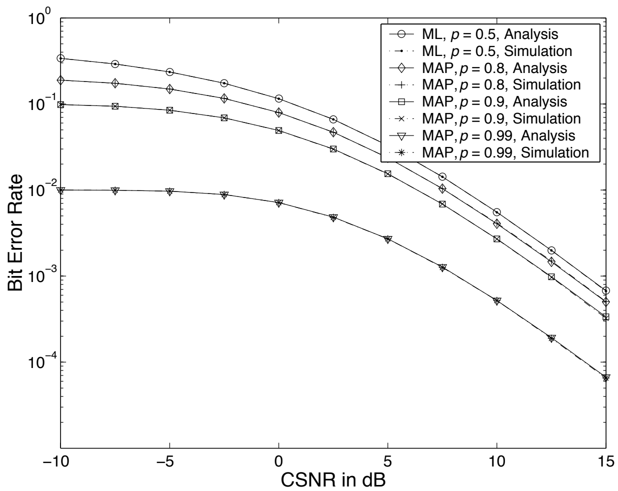

Figure 1: Results for BPSK signaling. $L_T = 2, L_R = 1$.

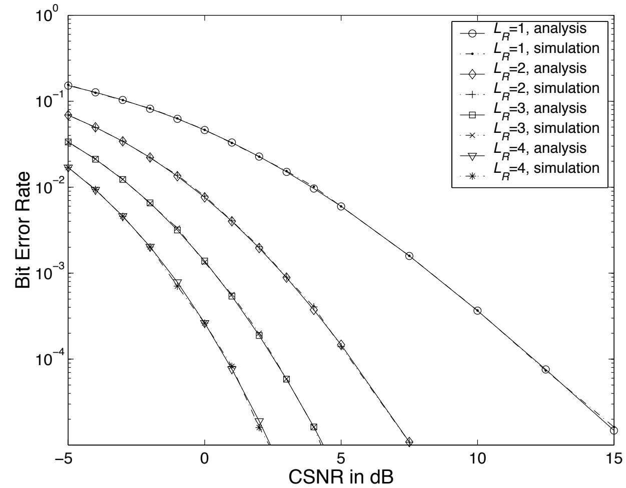

Figure 2: Results for BPSK signaling. $L_T = 3, p = 0.5$.

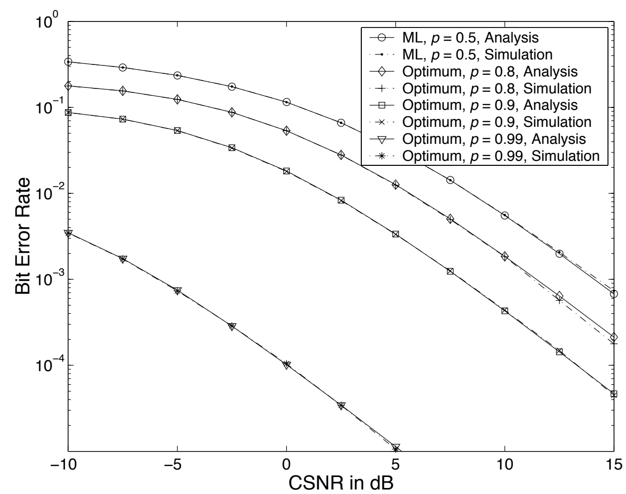

Figure 3: Results for the optimum binary antipodal signaling. $L_T = 2, L_R = 1$.

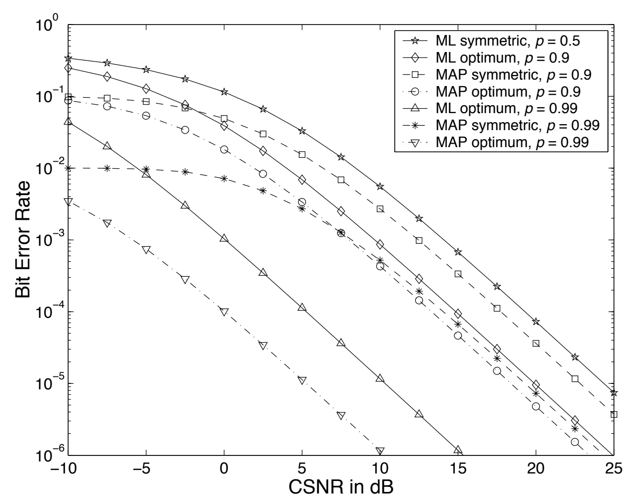

Figure 4: Comparison between BPSK and optimum signaling schemes. $L_T = 2, L_R = 1$.

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