Exact Pairwise Error Probability of Space-Time Codes under MAP Decoding with Applications*

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Abstract

We study the maximum *a posteriori* (MAP) decoding of memoryless non-uniform sources over multiple-antenna channels. Our model is general enough to include space-time coding, BLAST architectures, and single-transmit multi-receive antenna systems which employ maximal receive combining (MRC) and any kind of channel coding. We derive a closed-form single-letter expression for the codeword pairwise error probability (PEP) of general multi-antenna codes using moment generating function and Laplace transform arguments. We then utilize this result to design space-time linear dispersion codes which are optimized for the source distribution. Two codes are given here which outperform V-BLAST by 9.3 dB at a frame error rate (FER) of 2×10^{-3} with uniform i.i.d. input and BPSK signaling and by 8 dB for a non-uniform binary source with $p_0 \triangleq P(0) = 0.9$, MAP decoding, and Q-PSK signaling. The issue of bit-to-signal mapping is also studied. For a system with 1 transmit and 2 receive antennas which uses trellis-coded modulation with 16-QAM signaling, we observe that a better than Gray mapping outperforms Gray mapping by 1.0 dB when MAP decoding is used for transmitting binary sources with $p_0 = 0.9$.

1 Introduction

Ideally, a lossless source coder would compress data into an independent, identically distributed (i.i.d.) bit-stream at a rate equal to the entropy rate of the source (for sufficiently long blocklengths). However, most source coding methods are not ideal; hence there exists a residual redundancy (in the form of non-uniform distribution or memory) at their output which will be present at the input of the channel encoder. For example, in facsimile and medical imagery, the probability of having a "0" (as opposed to a "1") in the resulting bit-stream can be as high as 85% on average. A similar non-uniformity exists at the output of CELP speech vocoders (see, e.g., [1]). Another example is the bit-stream at the output of vector quantizers with moderate blocklengths.

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In this paper, we study how exploiting the source non-uniformity at the transmitter and/or the receiver can improve the performance of multi-antenna systems. Our contribution is three-fold. First, we derive the maximum *a posteriori* (MAP) decoding rule for multi-antenna codewords. We then derive a closed-form single-letter expression for the codeword pairwise error probability (PEP) of general multi-antenna codes (including space-time and BLAST codes as well as coded/uncoded maximal receive combining (MRC)) under MAP decoding. For maximum likelihood (ML) decoding, the main challenge in finding the PEPs of interest under fading is to average $Q(\sqrt{X})$ where $Q(\cdot)$ is the Gaussian error integral and X is a non-negative random variable. A closed-form expression for the codeword PEP of space-time codes of arbitrary structure under slow Rayleigh fading and ML decoding is derived in [8]. The derivation is based on an alternate formula for the $Q(\cdot)$ function in [3], which only works for non-negative arguments. As will be seen in the sequel, computing the PEP between a pair of MAP decoded codewords requires finding the expected value of $Q(\sqrt{X} + \lambda/\sqrt{X})$, where λ is a real (positive or negative) number, which is more involved than the ML decoding case. We make eigen decomposition and Laplace transform arguments to derive the above PEP. We then show that there can be a large gain in performing MAP decoding as compared with ML decoding.

Finally, we apply our results to two scenarios. First, we find space-time linear dispersion codes under both ML and MAP decoding. Unlike [5] – where the code design criterion is to maximize the mutual information between the channel input and output – we opt to minimize the union upper bound on the block error rate of the code. We show that even for a simple dual-transmit single-receive system with BPSK modulation, gains up to 11.2 dB can be obtained over V-BLAST for a uniform i.i.d. source at a bit error rate (BER) of 10^{-3}. Second, we address bit-to-signal mappings which take the input non-uniformity into account to minimize the union upper bound on the frame error rate (FER) of a system which uses trellis coded modulation (TCM) in a single-transmit multiple-receive antenna setup with MRC. We observe that the gains with better than Gray mappings can be significant if the source has non-uniform distribution. For example, in a TCM-based MRC system with 2 receive antennas and $p_0 \triangleq P(b = 0) = 0.9$ (where b is a data bit), at BER = 10^{-4}, a channel signal-to-noise ratio (CSNR) gain of 1.6 dB can be obtained through MAP decoding (instead of ML decoding) and an additional gain of 0.8 dB can be achieved using a signal mapping which is carefully designed (hence a total gain of 2.4 dB over Gray mapping and ML decoding).

The rest of this paper is organized as follows. Section 2 describes the multi-input multi-output (MIMO) channel model and formulates the MAP decoding rule. The exact codeword PEP is derived in Section 3. Applications to linear dispersion code design and bit-to-signal mapping for TCM-based MRC systems are presented in Section 4. Section 5 presents numerical results and the paper is concluded in Section 6.

2 System Model and The MAP Decoding Rule

The MIMO communication system considered here employs K transmit and L receive antennas. The input to the system is a stream of i.i.d. data bits $\{b_i\}$ which can have non-uniform distribution. The baseband constellation points are denoted by $\{c_{(k)}\}_{k=1}^{2p}$ where p is a positive integer. We will assume that $E\{|c_{(k)}|^2\} = 1$. We assume that every block of input symbols is encoded into a codeword matrix $\boldsymbol{S} = (\boldsymbol{s}_1, \boldsymbol{s}_2, ..., \boldsymbol{s}_w)$, where $\boldsymbol{s}_t = (s_{1,t}, s_{2,t}, ..., s_{K,t})^T$ is simultaneously transmitted, w is the codeword length in symbol periods, and ^{T} denotes transposition.[1] The channel is assumed to be Rayleigh flat fading, so that the complex path gain from transmit antenna i to receive antenna j, denoted by $H_{j,i}$, has a zero-mean unit-variance complex Gaussian distribution, denoted by $\mathcal{CN}(0,1)$, with i.i.d. real and imaginary parts. We assume that the receiver, but not the transmitter, has perfect knowledge of the path gains. Moreover, we assume that the channel is quasi-static, meaning that the path gains remain constant during a codeword transmission, but vary in an i.i.d. fashion from one codeword interval to another. The additive noise at the j^{th} receive antenna at time t, $N_{j,t}$, is assumed to be $\mathcal{CN}(0,1)$ distributed with i.i.d. real and imaginary parts. We will assume that the input, fading coefficients, and channel noise are independent from each other.

Based on the above, for a CSNR of γ_s at each receive branch and at time t, the signal at receive antenna j can be written as $R_{j,t} = \sqrt{\frac{\gamma_s}{K}} \sum_{i=1}^{K} H_{j,i} s_{i,t} + N_{j,t}$, or in matrix form,

$$\boldsymbol{r}_t = \sqrt{\frac{\gamma_s}{K}} \boldsymbol{H} \boldsymbol{s}_t + \boldsymbol{n}_t, \tag{1}$$

where $\boldsymbol{r}_t = (R_{1,t}, R_{2,t}, ..., R_{L,t})^T$, $\boldsymbol{H} = \{H_{j,i}\}$, and $\boldsymbol{n}_t = (N_{1,t}, N_{2,t}, ..., N_{L,t})^T$.

Let us denote the received signals corresponding to $\boldsymbol{S}$ by $\boldsymbol{R} = (\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_w)$ and the *a priori* probability of codeword $\boldsymbol{S}$ by $p(\boldsymbol{S})$. Assuming that perfect channel state information is available, in MAP decoding one aims to maximize $P(\boldsymbol{S}|\boldsymbol{R}, \boldsymbol{H})$ over the codebook. The MAP decoding rule is hence given by

$$\begin{aligned}
\arg\max_{\boldsymbol{S}} \; P\{\boldsymbol{S}|\boldsymbol{R}, \boldsymbol{H}\} &= \arg\max_{\boldsymbol{S}} \; P\{\boldsymbol{R}|\boldsymbol{S}, \boldsymbol{H}\} p(\boldsymbol{S}) \\
&= \arg\max_{\boldsymbol{S}} \; P\left\{ \boldsymbol{R} - \sqrt{\frac{\gamma_s}{K}} \boldsymbol{H}\boldsymbol{S} \,\middle|\, \boldsymbol{S}, \boldsymbol{H} \right\} p(\boldsymbol{S}) \\
&= \arg\max_{\boldsymbol{S}} \; p(\boldsymbol{S}) \frac{1}{\sqrt{2\pi}} \prod_{j,t} \exp\left\{ - \left| R_{j,t} - \sqrt{\frac{\gamma_s}{K}} \sum_i H_{j,i} s_{i,t} \right|^2 \right\} \\
&= \arg\min_{\boldsymbol{S}} \; \left\{ -\ln\left(p(\boldsymbol{S})\right) + \sum_t \sum_j \left| R_{j,t} - \sqrt{\frac{\gamma_s}{K}} \sum_i H_{j,i} s_{i,t} \right|^2 \right\}. \quad (2)
\end{aligned}$$

3 The Codeword Pairwise Error Probability under MAP Decoding

The codeword PEP between $\boldsymbol{S}$ and $\hat{\boldsymbol{S}}$ is defined as the probability that $\boldsymbol{S}$ has a larger MAP metric in (2) than $\hat{\boldsymbol{S}}$ given that $\boldsymbol{S}$ is transmitted. Therefore,

$$P(\boldsymbol{S} \to \hat{\boldsymbol{S}}|\boldsymbol{H}) = P\left\{ \sum_t \sum_j \left| -\sqrt{\frac{\gamma_s}{K}} \sum_i H_{j,i} d_{i,t} + N_{j,t} \right|^2 - \ln p(\hat{\boldsymbol{S}}) < \sum_t \sum_j |N_{j,t}|^2 - \ln p(\boldsymbol{S}) \right\},$$

where $d_{i,t} = s_{i,t} - \hat{s}_{i,t}$. The codeword PEP is therefore equal to

$$\begin{aligned}
P(\boldsymbol{S} \to \hat{\boldsymbol{S}}) &= E_{\boldsymbol{H}} \left\{ P\left\{ \sqrt{2} \sum_t \sum_j \langle N_{j,t}, \sum_i H_{j,i} d_{i,t} \rangle > \frac{1}{\sqrt{2}} \Delta_{\boldsymbol{S},\hat{\boldsymbol{S}}} + \frac{\sqrt{2}}{\Delta_{\boldsymbol{S},\hat{\boldsymbol{S}}}} \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}} \right\} \right\} \\
&= E_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}} \left\{ Q\left(\sqrt{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}} + \frac{1}{\sqrt{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}} \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}} \right) \right\}, \qquad (3)
\end{aligned}$$

[1] Note that one can interpret w as the frame length and hence this model is general enough to include space-time trellis codes such as those in [10].

where

$$\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}} = \frac{\gamma_s}{K}\sum_t\sum_j\left|\sum_i H_{j,i}d_{i,t}\right|^2 \quad \text{and} \quad \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}} = \frac{1}{2}\ln\frac{p(\boldsymbol{S})}{p(\hat{\boldsymbol{S}})}.$$

We note that $\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}$ can take on negative values. Hence, one cannot use the alternate formula of the $Q(\cdot)$ function in [3], as is done in [8] for ML decoding. We therefore proceed as follows. To compute the expectation (3) in closed-form, we determine the probability density function (pdf) of $\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}$, convert (3) into a linear combination of the derivatives of $\mathcal{L}\left\{Q\left(\sqrt{x}\right)\right\}$, where $\mathcal{L}(\cdot)$ is the Laplace transform operator, and then evaluate these derivatives. First, we note that

$$\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}} = \frac{\gamma_s}{K}\sum_{t=1}^{w}\sum_{j=1}^{L}\left|\sum_{i=1}^{K} H_{j,i}d_{i,t}\right|^2 = \frac{\gamma_s}{K}\sum_{j=1}^{L}\boldsymbol{h}_j^{\dagger}\mathbf{U}\boldsymbol{h}_j, \tag{4}$$

where $u_{k,i} = \sum_t d_{i,t}d^*_{k,t}$, and $\boldsymbol{h}_j$ is the transpose of the j^{th} row of $\boldsymbol{H}$. Since $\boldsymbol{U}$ is Hermitian (i.e., $\boldsymbol{U}^{\dagger} = \boldsymbol{U}$, where † represents complex conjugate transposition) and non-negative definite, it can be decomposed as

$$\boldsymbol{U} = \boldsymbol{V}^{\dagger}\boldsymbol{D}\boldsymbol{V}, \tag{5}$$

where $\boldsymbol{D}$ is a non-negative definite diagonal matrix having the eigenvalues of $\boldsymbol{U}$ on its main diagonal. $\boldsymbol{V}$ is a unitary matrix (i.e., $\boldsymbol{V}^{\dagger}\boldsymbol{V} = \mathbf{I}_K$) and its columns are the unit-norm eigenvectors of $\boldsymbol{U}$. Substituting (5) in (4), we have

$$\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}} = \frac{\gamma_s}{2K}\sum_{j=1}^{L}\boldsymbol{h}_j^{\dagger}\boldsymbol{V}^{\dagger}\boldsymbol{D}\boldsymbol{V}\boldsymbol{h}_j = \frac{\gamma_s}{2K}\sum_{j=1}^{L}\boldsymbol{x}_j^{\dagger}\boldsymbol{D}\boldsymbol{x}_j = \sum_{j=1}^{L}\sum_{i=1}^{K}\frac{\gamma_s}{2K}\lambda_i|X_{ji}|^2, \tag{6}$$

where $\boldsymbol{x}_j = \boldsymbol{V}\boldsymbol{h}_j$, X_{ji} is the i^{th} element of $\boldsymbol{x}_j$, and $\lambda_i = \boldsymbol{D}_{i,i}$. As $\boldsymbol{V}$ is unitary, the entries of $\boldsymbol{x}_j$ are i.i.d. $\mathcal{CN}(0,1)$ and the moment generating function (MGF) of $\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}$ is

$$\Phi_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}(s) = \prod_{k=1}^{Z}\frac{1}{\left(1-\lambda_k\frac{\gamma_s}{2K}s\right)^{Ln_k}}, \tag{7}$$

where Z is the number of distinct non-zero λ_k's each of multiplicity n_k (with appropriate re-ordering of the eigenvalues). The pdf of a random variable Θ is the inverse Laplace transform ($\mathcal{L}^{-1}$) of $\Phi_\Theta(-s)$. In order to find $\mathcal{L}^{-1}\left\{\Phi_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}(-s)\right\}$, we convert (7) into a sum and then use the linearity of the Laplace transform. Letting $p_k = \frac{2K}{\gamma_s\lambda_k}$, we can write the partial-fraction expansion of (7) as

$$\Phi_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}(-s) = \prod_{k=1}^{Z}\frac{1}{(1+\frac{s}{p_k})^{Ln_k}} = \prod_{k=1}^{Z}\frac{p_k^{Ln_k}}{(s+p_k)^{Ln_k}} = \sum_{k=1}^{Z}\sum_{i=0}^{Ln_k-1}\frac{\alpha_{i+1,k}}{(s+p_k)^{i+1}}, \tag{8}$$

where

$$\alpha_{Ln_k-i,k} = \frac{1}{i!}\left\{\frac{d^i}{ds^i}\left[(s+p_k)^{Ln_k}\Phi_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}(-s)\right]\right\}\Bigg|_{s=p_k}, \qquad i = 0,\ldots,Ln_k-1. \tag{9}$$

Taking the inverse Laplace transform of the right hand side of (8), we have

$$f_{\frac{1}{2}\Delta^2_{\boldsymbol{S},\hat{\boldsymbol{S}}}}(x) = \sum_{k=1}^{Z}\sum_{i=0}^{Ln_k-1}\frac{\alpha_{i+1,k}}{i!}x^i e^{-\frac{2K}{\gamma_s\lambda_k}x}, \qquad x \geq 0. \tag{10}$$

The next step is simply using (10) to evaluate (3). This yields

$$P(\boldsymbol{S} \to \hat{\boldsymbol{S}}) = \sum_{k=1}^{Z} \sum_{i=1}^{Ln_k} \frac{\alpha_{i,k}}{(i-1)!} \int_0^\infty y^{i-1} e^{-\frac{2K}{\gamma_s \lambda_k} y} Q\left(\sqrt{y} + \frac{\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}}{\sqrt{y}}\right) dy. \tag{11}$$

We note that the integral in (11) is the Laplace transform of $y^{i-1} Q\left(\sqrt{y} + \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}/\sqrt{y}\right)$ evaluated at $s = \delta_k^{-2} \triangleq 2K/\gamma_s\lambda_k$. We know that if $f(t)$ and $F(s)$ are Laplace transform pairs ($F(s) = \mathcal{L}\{f(t)\}$), so are $t^n f(t)$ and $(-1)^n \frac{d^n}{ds^n} F(s)$. Therefore, we need to find the $i-1^{\text{st}}$ derivative of $\mathcal{L}\left\{Q\left(\sqrt{y} + \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}/\sqrt{y}\right)\right\}$. It can be shown that $F_{\text{MAP}}(s)$, the Laplace transform of $Q\left(\sqrt{y} + \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}/\sqrt{y}\right)$, is equal to

$$F_{\text{MAP}}(s) = \frac{1 - \text{sgn}(\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}})}{2s} - \frac{1}{2}\left(\frac{1}{s\sqrt{2s+1}} + \frac{\text{sgn}(\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}})}{s}\right) e^{-(\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}} + |\Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}|\sqrt{2s+1})}, \tag{12}$$

where $\text{sgn}(x) = \frac{|x|}{x}$ if $x \neq 0$ and 0 otherwise. The term in the sum in (11) is simply

$$\frac{\alpha_{i,k}}{(i-1)!}(-1)^{i-1} \frac{d^{i-1}}{ds^{i-1}} F_{\text{MAP}}(s) \triangleq \alpha_{i,k}\delta_k^2 \pi(i, \delta_k, \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}).$$

We used the Leibniz's formula for the i^{th} derivative of a multiplication as well as induction to find the i^{th} derivative of (12). The result is the following expression for the exact codeword PEP of MAP decoded space-time codes

$$P(\boldsymbol{S} \to \hat{\boldsymbol{S}}) = \sum_{k=1}^{Z} \sum_{i=1}^{Ln_k} \delta_k^{2i} \alpha_{i,k} \pi(i, \delta_k, \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}), \tag{13}$$

where

$$\begin{aligned} \pi(n,\delta,\lambda) \quad = \quad & \frac{1-\text{sgn}(\lambda)}{2} - \frac{1}{2} e^{-\left(\lambda + |\lambda|\sqrt{2\delta^{-2}+1}\right)} \times \\ & \sum_{k=0}^{n-1} \frac{(-1)^{n+k-1}}{(2+\delta^2)^{n-k-1}} \left(-\text{sgn}(\lambda) + \frac{\delta}{\sqrt{2+\delta^2}} \sum_{m=0}^{k} \binom{2m}{m} \frac{1}{(2\delta^2+4)^m}\right) \times \\ & \sum_{l=1}^{n-k-1} \frac{|\lambda|^l (\delta^2+2)^{l/2}}{l!\delta^l} \sum_{p=0}^{l-1} \binom{l}{p} (-1)^{l+p} \prod_{q=0}^{n-k-2} (l-p-2q) \end{aligned} \tag{14}$$

where $\sum_{i=L}^{U} z_i \triangleq 1$ if $L > U$.

4 Applications

4.1 Linear Dispersion Code Design for MAP Decoding

Linear dispersion codes (introduced in [5]) constitute an important class of space-time block codes. Every entry of an LD codeword is a weighted sum of the baseband signals with the weights chosen such that the mutual information between the channel input and output is maximized given the number of transmit and receive antennas. An LD codeword is written as

$$\boldsymbol{S} = \sum_{m=1}^{M} (\alpha_m \boldsymbol{A}_m + j\beta_m \boldsymbol{B}_m),$$

where $\boldsymbol{A}_m$ and $\boldsymbol{B}_m$ are $K \times w$ matrices (similar to [5], we assume that $\boldsymbol{A}_m$ and $\boldsymbol{B}_m$ have real entries), $c_m = \alpha_m + j\beta_m$ is a symbol to be encoded ($j = \sqrt{-1}$), and M is the block length in symbols (i.e., the number of data symbols to be encoded at a time).

Instead of maximizing the mutual information, here we opt to design LD codes via minimizing the union upper bound on the frame (block) error rate which can be computed using the codeword PEP formula given in (13). As mentioned in [5], calculating the codeword PEP at high CSNRs can face numerical problems, but we have noticed that the codes are not sensitive to the design CSNR; i.e., those which are designed at low CSNRs have also a good performance at high CSNR. Therefore, our design CSNR is often set at 5 to 10 dB.

Our design method is as follows: to guarantee maximum throughput and to make sure that the performance is always better than V-BLAST, we begin with the A_m and B_m matrices which correspond to V-BLAST and are given by [4, 5]

$$\boldsymbol{A}_{K(\tau-1)+k} = \boldsymbol{B}_{K(\tau-1)+k} = \boldsymbol{c}_\tau \boldsymbol{d}_k^T, \quad \tau = 1, ..., w, k = 1, ..., K,$$

where $\boldsymbol{c}_\tau$ and $\boldsymbol{d}_k$ are w-dimensional and k-dimensional column all-zero vectors except for a 1 in the τ^{th} and k^{th} entries, respectively. We then improve the code by adding zero-mean Gaussian noise to the $\boldsymbol{A}_m$ and $\boldsymbol{B}_m$ matrices, normalizing the $\boldsymbol{A}_m$ and $\boldsymbol{B}_m$ matrices to satisfy the power constraint according to [5, equation(18)] which is given by

$$\sum_m (\text{tr}\boldsymbol{A}_m^\dagger \boldsymbol{A}_m + \text{tr}\boldsymbol{B}_m^\dagger \boldsymbol{B}_m) = 2wK, \tag{15}$$

and updating the code if the new FER union bound decreases. We have chosen the variance of the additive noise to decrease according to $\sigma_n^2 = 0.25\,(1 - i/I_{\text{max}})^3\,, i = 1, 2, ..., I_{\text{max}}$, where i is the iteration number. This regime is chosen following [12] due to its fast convergence rate and good results. Note that this new code is still a linear dispersion code. Therefore:

1. Similar to [5], we have noticed that the performance of the resulting codes is not sensitive to the design CSNR. Therefore, to avoid numerical problems resulting from the addition of very small numbers, we set the design CSNR at 5 to 10 dB, depending on the number of antennas.

2. The design criterion in [5] is to maximize mutual information between channel input and output assuming that the real and imaginary parts of the signal set have $\mathcal{N}(0, \frac{1}{2})$ distribution, which may be far from the particular signaling scheme and non-uniform distribution to be used. In our method, we optimize the code for the particular signaling scheme and prior probabilities which are going to be used. Obviously, the design method works as well with the assumption of having $\mathcal{N}(0, 1)$ distribution for the signals.

3. The power constraint (15) can be made more restrictive. For example, one could use $\boldsymbol{A}_m^\dagger \boldsymbol{A}_m = \boldsymbol{B}_m^\dagger \boldsymbol{B}_m = \frac{w}{M}\mathbf{I}_K$ for $m = 1, ..., M$. It is noted in [5] that this power constraint generally leads to lower error rates, but we have used (15) in our design to "relax" the condition as much as possible and let the search algorithm converge to any local minimum which satisfies the power constraint.

4. Obviously, the search method is random and may converge to a local minimum. The variance of the additive noise is large at the initial loops to allow large improvements, but it reduces with the iterations to allow convergence and small refinement. We have observed that many small changes are made at lower noise variances.

5. In order to have the possibility of finding better minima, we run the algorithm twice with the second round initialized with the results of the first. This allows "escaping" from a bad local minimum at the beginning of the second round, when the variance of the additive noise is large.

4.2 Trellis-Coded Modulation and Maximal-Receive Combining

As already mentioned, the results of Section 3 are valid for single-input multi-output systems as well. This subsection presents an application of the MAP decoding rule in (2). In particular, we describe how (13) can be used to find TCM encoder and signal mappings which, when used with the MAP decoding rule in (2), will reduce the frame error rate (FER) and BER of a TCM-coded MRC system. The discussion is made for Ungerboeck-type systematic convolutional encoder structures with feedback (see [11, Fig. 10]).

First, we note that for a single transmit-antenna system, we have

$$\alpha_i = \begin{cases} 0 & \text{for } i = 1, 2, ..., L-1 \\ \left(\frac{2}{\lambda\gamma_s}\right)^L & \text{for } i = L \end{cases}$$

and hence we have

$$P(\boldsymbol{S} \to \hat{\boldsymbol{S}}) = \pi(L, \delta, \Lambda_{\boldsymbol{S},\hat{\boldsymbol{S}}}), \tag{16}$$

where $\delta^2 = \gamma_s/2\sum_t |d_t|^2$. In [7], a method is presented to approximate the union upper bound for the FER of trellis codes over AWGN channels under MAP decoding for asymmetric binary Markov sources. Using (16) and the method in [7], one can search for good trellis codes for the multiple receive-antenna system considered here.

5 Numerical Results

5.1 Linear Dispersion Code Design

Figure 1 demonstrates the performance of our LD code search method for uniform and non-uniform (with $p_0 = 0.9$) binary i.i.d. sources, respectively, for a dual-transmit single-receive antenna system. As mentioned in Subsection 4.1, we start from a V-BLAST structure, so $M = 4$. It is observed that the new code significantly outperforms V-BLAST and its gain over V-BLAST continues to grow as the CSNR increases. We believe that this behavior is due to the larger diversity order of LD codes over V-BLAST, as the LD codes send each signal over all transmit antennas while in V-BLAST each signal is sent once from only one transmit antenna, so it experiences only one fading coefficient.

We first used BPSK modulation with uniform input. The new code, denoted by LDC1 and designed for CSNR = 5 dB, is specified by

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.74342773 & -0.08692446 \\ 0.40936511 & 0.50797666 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} -0.36902827 & 0.54303939 \\ 0.75611440 & -0.00025825 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0.30229420 & 0.83952155 \\ -0.45246174 & 0.09078508 \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} -0.47303478 & -0.03771225 \\ -0.21804135 & 0.85410106 \end{bmatrix},$$

and $\boldsymbol{B}_i = \boldsymbol{0}_{2\times 2}, i = 1, ..., 4$ (because the signals are real). At FER = 2×10^{-3}, the CSNR gain in using the LDC1 code is 9.3 dB over V-BLAST and 3 dB over the LD code of [5, eq. (31)]. This gain is 7.1 dB at BER = 10^{-3} over V-BLAST and 6 dB over the LD code of [5, eq. (31)].

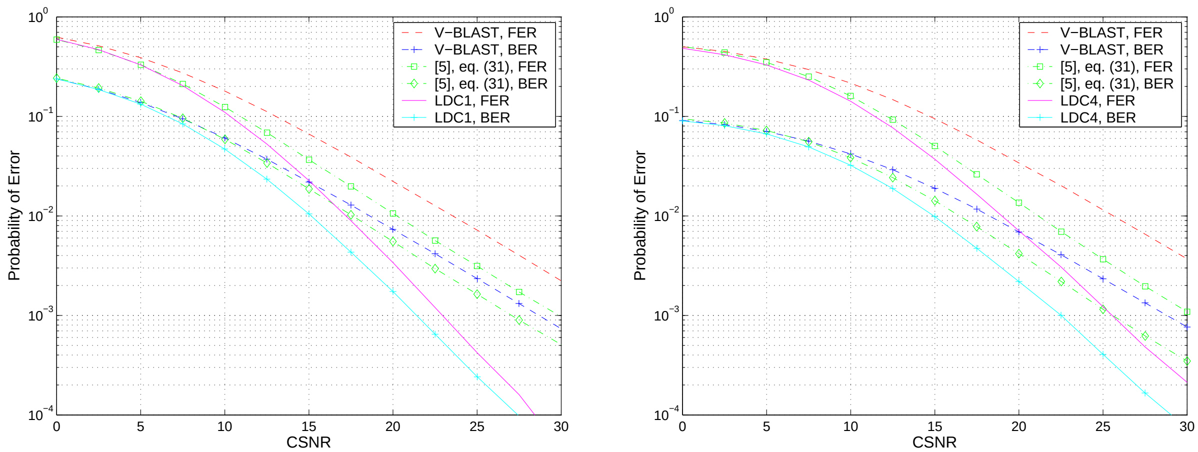

Figure 1: Comparison among V-BLAST, the LD code of [5, eq. (31)], and the new code; $K = 2, L = 1$. Left: BPSK modulation and ML decoding with $p_0 = 0.5$, Right: Q-PSK modulation and MAP decoding with $p_0 = 0.9$.

A non-uniform source is considered in Figure 1 (right) with Q-PSK modulation and the new LDC4 code, which is designed for CSNR = 5 dB, is found as

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.53118783 & -0.61393180 \\ 0.68193007 & 0.59598146 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} -0.82682148 & -0.20327547 \\ 0.30826739 & -0.79915328 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0.78416847 & 0.62394886 \\ 0.10937077 & -0.58952535 \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} -0.05075264 & 0.60912624 \\ -0.76206220 & 0.51194349 \end{bmatrix},$$

and

$$\boldsymbol{B}_1 = \begin{bmatrix} 0.19613700 & 0.49555234 \\ 0.30362879 & -0.28257270 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0.27837874 & 0.52828956 \\ 0.37751169 & 0.04153397 \end{bmatrix},$$

$$\boldsymbol{B}_3 = \begin{bmatrix} 0.39168615 & 0.03146527 \\ 0.49255746 & -0.57514978 \end{bmatrix}, \quad \boldsymbol{B}_4 = \begin{bmatrix} 0.67666998 & 0.13030253 \\ -0.16359026 & -0.54365423 \end{bmatrix}.$$

At FER = 4×10^{-3}, the CSNR gain for LDC4 is 8 dB over V-BLAST and 3.1 dB over the code of [5, eq. (31)]. At BER = 10^{-3}, the CSNR gain is 6.2 dB over V-BLAST and 3.1 dB over the code of [5, eq. (31)], respectively. Similar gains are observed for other source distributions and numbers of transmit or receive antennas.

5.2 Trellis-Coded Modulation with Maximal-Receive Combining

The system considered here has one transmit and two receive antennas. The frame length is 120 bits and the test is repeated 200000 times. A frame error is counted when the decoded and transmitted symbol streams do not exactly match (i.e., even if one symbol is different). As outlined in Section 4.2, we optimize two TCM systems using MRC via minimizing the union upper bound expression on the FER which is in terms of the PEP formula in (16). The first system is optimized in terms of the convolutional encoder structure for a fixed bit-to-signal mapping; while the second system is optimized with respect to the bit-to-signal mapping for a fixed convolutional encoder structure.

We first consider rate-$2/3$ 8-state 8-PSK trellis codes. Figure 2 shows that while there is a 4 dB gain in CSNR at BER = 10^{-3} in MAP decoding over ML decoding, the error rates of

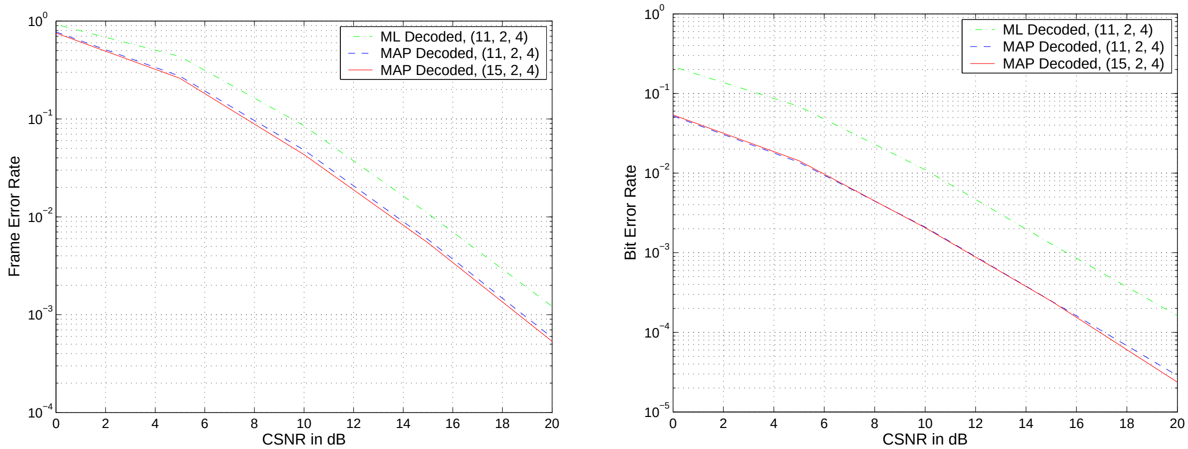

Figure 2: Performance of the rate $2/3$ 8-state 8-PSK TCM-code with $L = 2$ receive antennas and natural binary code mapping. The input is an i.i.d. bit-stream with $p_0 = 0.9$. The encoder structure is specified by (h_0, h_1, h_2) in octal, as in [11].

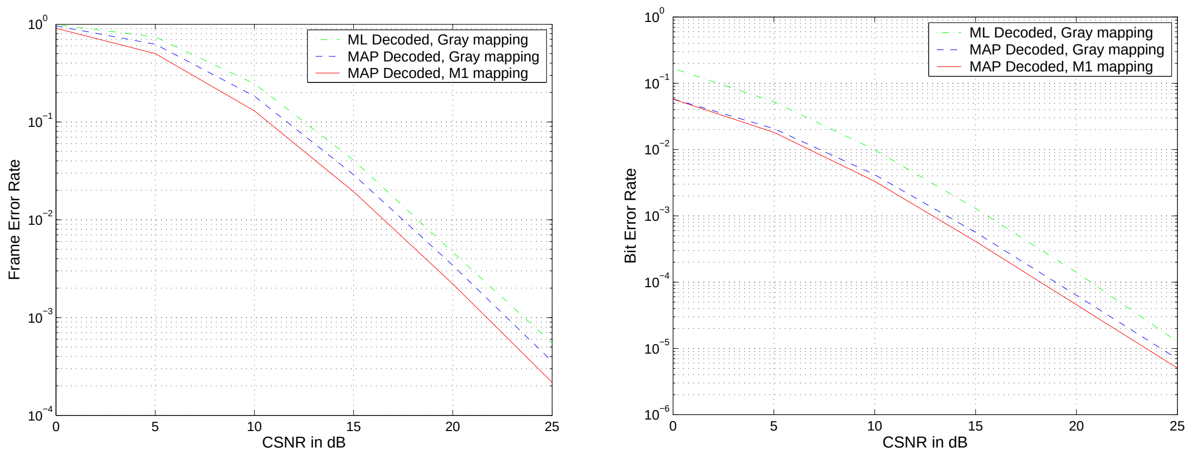

Figure 3: Comparison between the M1 and Gray mappings for the rate $3/4$ 8-state 16-QAM TCM-code with $L = 2$ receive antennas. The input is an i.i.d. bit-stream with $p_0 = 0.9$.

the trellis code with the optimized (15, 2, 4) encoder are not significantly less than the original (11, 2, 4) system (from [6, Page 174]).

Figure 3 demonstrates the performance of a rate-$3/4$ 8-state 16-QAM trellis coded system. We optimize the bit-to-signal mapping using the symbol PEP formula in [2] and the guidelines of [9], which result in the M1 mapping of [9, Figure 8]. We compare the M1 and Gray mapped systems for the same encoder structure specified by $(h_0, h_1, h_2) = (11, 2, 4)$. It is observed that, since the M1 mapping is more energy-efficient, it achieves a 1 dB CSNR gain over Gray mapping with MAP decoding at FER = 10^{-3}, and an additional 0.8 dB CSNR gain over ML decoding; i.e., 1.8 dB gain in CSNR over the conventional which use Gray mapping and ML decoding.

6 Conclusion

In this paper, we addressed the maximum *a posteriori* decoding of non-uniform i.i.d. sources in a multiple-antenna setting. We derived the MAP decoding rule and a closed-form single-letter expression for the codeword PEP of general multi-antenna codes. We also explored some applications of our two results. We designed space-time LD codes which were optimized for the source distribution. Two typical codes were given which outperformed V-BLAST by a wide margin. We also addressed the issue of bit-to-signal mapping in TCM-coded MRC systems.

Extensions of this work may include optimization of bit-to-signal mapping for space-time coded channels with non-uniform input and finding the codeword PEP for the case where the source has memory in addition to non-uniform distribution.

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