Error Analysis for Nonuniform Signaling Over Rayleigh Fading Channels

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***Abstract*—We investigate the error performance of a communication system where a nonuniform memoryless binary source is transmitted via Gray-mapped M-ary phase-shift keying or quadrature amplitude modulation over memoryless Rayleigh fading channels, and demodulated via optimal maximum *a posteriori* detection. Using recently derived upper and lower bounds on the probability of a general union of events, which are tight and can be efficiently computed, the system symbol-error (P_s) and bit-error (P_b) rates are evaluated for a wide range of channel conditions. Since for nonuniform signaling, Gray mapping is not necessarily optimal for minimizing P_s or P_b (as was recently shown by Takahara *et al.*), we also evaluate the system performance under the map obtained by Takahara *et al.* and compare it with a Gray-mapped system.**

***Index Terms*—Bit and symbol error rates, Gray mappings, lower and upper bounds, maximum *a posteriori* (MAP) decoding, nonuniform sources, phase-shift keying (PSK) and quadrature amplitude modulation (QAM) modulations, probability of a union, Rayleigh fading channels.**

I. Introduction

In recent work (e.g., [3], [11], [12], and [14]), important efforts have been devoted to the error analysis of the transmission of data sources over fading channels. Motivated by the fact that many compressed or uncompressed data sources, such as image or speech signals, are nonuniformly distributed (e.g., [1], [7], etc.), we focus our study on the error performance when nonuniform M-ary signals are transmitted over Rayleigh fading channels. If s_u is the transmitted signal, the symbol-error rate (SER) (P_s) under maximum *a posteriori* (MAP) decoding can be written as

$$P_s = \sum_{u=1}^{N} P(\epsilon|s_u)P(s_u) = \sum_{u=1}^{N} P\left(\bigcup_{i \neq u} \epsilon_{ui} \middle| s_u\right) P(s_u) \quad (1)$$

where $P(\epsilon|s_u)$ is the conditional probability of error given that s_u was sent, and ϵ_{ui} represents the event that s_i has a higher MAP metric than s_u, where $s_i \neq s_u$. However, the probability of a union of events is often difficult to compute explicitly, since, in general, it requires taking into account all combinations of event intersections. A common method to address this issue is to employ the standard union upper bound (or variations of it) in estimating the error performance. Although the union bound generally results in relatively simple expressions (depending on the communication system under analysis), it is often loose for a wide range of channel-noise conditions, particularly for moderate or low signal-to-noise ratios (SNRs). An attractive approach is, therefore, to investigate more sophisticated bounds that yield accurate error estimates. Three such bounds on the probability of a finite union of arbitrary events $P(\bigcup_{i=1}^{N} A_i)$ were recently studied in [8] and [9]: a lower bound (the Kuai–Alajaji–Takahara (KAT) bound) [8], a practical algorithmic stepwise lower bound [9] originating from Kounias [6], and a greedy algorithmic implementation [9] of an upper bound due to Hunter [5]. In this letter, we apply these bounds, which are only expressed in terms of the individual event probabilities $P(A_i)$ and the pairwise event probabilities $P(A_i \cap A_j)$, to estimate the performance of nonuniform signaling under Gray mapping over memoryless Rayleigh fading channels. Furthermore, it was recently shown in [15] that Gray mapping is not necessarily optimal for minimizing P_s or the bit-error rate (BER) P_b when nonuniform sources are sent over additive white Gaussian noise (AWGN) channels. It was indeed observed that an appropriately constructed mapping (the M_1 map) can perform significantly better than the Gray map (with gains as high as 3.5 dB in SNR for strongly biased sources). We thus investigate the error performance of our system under such a map. As a byproduct of our results, we also validate the choice of this map for the Rayleigh fading channel by noting that its performance is nearly identical to that of the optimal map, which minimizes the union upper bound of P_s while keeping a minimal average symbol energy. This paper is, in a sense, an extension of [9] and [15], where nonuniform signaling over AWGN channels was investigated.

The rest of the letter is organized as follows. The problem of bounding P_s and P_b for nonuniform signals transmitted over Rayleigh fading channels (with known channel-fading information at the receiver) used in conjunction with M-ary coherent phase-shift keying (PSK) or quadrature amplitude modulation (QAM) and symbol MAP decoding is investigated in Section II. The system performance is then evaluated for various PSK and QAM schemes in Section III. Finally, conclusions are stated in Section IV.

II. Nonuniform Signaling Over Rayleigh Channels

Consider a nonuniform independent and identically distributed (i.i.d.) binary source $\{X_i\}$ (with distribution $P\{X = 0\} = p$) which is grouped in blocks of $\log_2 M$ bits (we assume that M is a power of two). Each block is subsequently M-PSK or M-QAM modulated with mapping Gray or M_1 [15]. Then, the M-ary modulated signal sequence is transmitted over a Rayleigh fading channel, and is decoded via the optimal MAP criterion at the receiver. More specifically, if one of the M two-dimensional (2-D) signals $s_1, s_2, \ldots, s_M$ is

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sent, say s_u, then the MAP decoder declares that s_i was sent, for $j = 1, 2, \ldots, M$ and $j \neq i$, if the MAP metric of s_i is bigger than the metric of s_j; i.e., $P(s_i|r) \geq P(s_j|r)$, where $r = \alpha s_u + n$ is the received 2-D vector signal, α is a Rayleigh distributed amplitude fading variable with second moment $2\sigma^2$ (i.e., α is the square root of the sum of two squared independent zero-mean Gaussian random variables, each with variance σ^2), and n is a 2-D noise vector with zero-mean uncorrelated Gaussian distributed components, each with variance $N_0/2$. We also assume that α, s_u, and n are independent from each other.

A. Symbol-Error Rate (SER)

In order to properly apply the bounds of [8] and [9] on (1), we need to determine the $P(\epsilon_{ui}|s_u)$ and $P(\epsilon_{ui} \cap \epsilon_{uj}|s_u)$ event probabilities. Assuming that the fading variable α can be estimated from the received signal without error, we can derive the conditional individual and pairwise error probabilities, given the channel fading and that s_u is sent

$$\begin{aligned} P(\epsilon_{ui}|\alpha, s_u) &\\ = \Pr\{f(r|s_i, \alpha)P(s_i) \geq f(r|s_u, \alpha)P(s_u)|\alpha, s_u\} &\\ = Q(\phi_{ui}(\alpha)) & \qquad (2) \end{aligned}$$

$$\begin{aligned} P(\epsilon_{ui} \cap \epsilon_{uj}|\alpha, s_u) &\\ = \Pr\{f(r|s_i, \alpha)P(s_i) \geq f(r|s_u, \alpha)P(s_u), &\\ f(r|s_j, \alpha)P(s_j) \geq f(r|s_u, \alpha)P(s_u)|\alpha, s_u\} &\\ = \Psi(\rho_{uij}, \phi_{ui}(\alpha), \phi_{uj}(\alpha)) & \qquad (3) \end{aligned}$$

where $f(r|\cdot,\cdot)$ is the conditional density function of the received signal r

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-t^2/2)\, dt \qquad (4)$$

$$\rho_{uij} = \frac{\langle s_i - s_u, s_j - s_u \rangle}{\|s_i - s_u\| \cdot \|s_j - s_u\|} \qquad (5)$$

$$\Psi(\rho, a, b) = \frac{1}{2\pi\sqrt{1-\rho^2}} \times \int_a^\infty \int_b^\infty \exp\left[-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right] dx\, dy \qquad (6)$$

$$\phi_{ui}(\alpha) = \frac{\alpha d_{ui}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0} \ln P(s_u)/P(s_i)}{2\alpha d_{ui}} \qquad (7)$$

$\langle \cdot, \cdot \rangle$ denotes the usual dot product, and $d_{ui} = \|s_i - s_u\|$ (where $\|\cdot\|$ is the Euclidean norm). Note that (6) is valid for $|\rho_{uij}| < 1$. When $|\rho_{uij}| = 1$, i.e., when signals s_u, s_i, and s_j are colinear (which may occur in QAM constellations), $\Psi(\rho_{uij}, \cdot, \cdot)$ in (3) reduces to the following expressions:

$$\Psi(1, \phi_{ui}(\alpha), \phi_{uj}(\alpha)) = Q(\max(\phi_{ui}(\alpha), \phi_{uj}(\alpha)))$$
$$\Psi(-1, \phi_{ui}(\alpha), \phi_{uj}(\alpha)) = [Q(\phi_{ui}(\alpha)) - Q(-\phi_{uj}(\alpha))]^+$$

where $[a]^+ = \max(0, a)$. Therefore

$$\begin{aligned} P(\epsilon_{ui}|s_u) &= E_\alpha[P(\epsilon_{ui}|\alpha, s_u)] \\ &= \int_0^\infty Q(\phi_{ui}(\alpha)) \frac{\alpha}{\sigma^2} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) d\alpha. \qquad (8) \end{aligned}$$

By integrating the right-hand side (RHS) of (8) by parts, we can show that

$$P(\epsilon_{ui}|s_u) = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\tau_{ui}}\right) \exp\left[-\frac{\omega_{ui}}{2}(1 + \tau_{ui})\right], & \text{if } \omega_{ui} \geq 0 \\ 1 - \frac{1}{2}\left(1 + \frac{1}{\tau_{ui}}\right) \exp\left[-\frac{\omega_{ui}}{2}(1 - \tau_{ui})\right], & \text{if } \omega_{ui} < 0 \end{cases} \qquad (9)$$

where $\omega_{ui} = \ln[P(s_u)/P(s_i)]$ and $\tau_{ui} = \sqrt{(\sigma^2 d_{ui}^2 + 2N_0)/(\sigma^2 d_{ui}^2)}$. Note that, as shown by (9), $P(\epsilon_{ui}|s_u)$ admits a closed-form expression, unlike the case when the channel is AWGN (with no fading). Furthermore, as expected, when the signaling is uniform $(p = 0.5)$, (9) reduces to the familiar expression of $P(\epsilon_{ui}|s_u)$ given, for example, in [14, eq. (5.72)].

For $\phi_{ui}(\alpha) \geq 0$ and $\phi_{uj}(\alpha) \geq 0$, we can use the result of [13] to write (6) as

$$\begin{aligned} \Psi(\rho, \phi_{ui}(\alpha), \phi_{uj}(\alpha)) &\\ = \frac{1}{2\pi} \int_0^{\vartheta(\phi_{ui}(\alpha)/\phi_{uj}(\alpha))} \exp\left[-\frac{\phi_{ui}(\alpha)^2}{2\sin^2\Phi}\right] d\Phi &\\ + \frac{1}{2\pi} \int_0^{\vartheta(\phi_{uj}(\alpha)/\phi_{ui}(\alpha))} \exp\left[-\frac{\phi_{uj}(\alpha)^2}{2\sin^2\Phi}\right] d\Phi & \qquad (10) \end{aligned}$$

where $\vartheta(x) = \tan^{-1}(x\sqrt{1-\rho^2}/(1-\rho x))$ for $x \geq 0, -1 < \rho < 1$.

If one or both of the second and the third arguments in the $\Psi(\cdot)$ function (6) are negative, the equalities shown in (11) at the bottom of the page can be used.

Therefore

$$\begin{aligned} P(\epsilon_{ui} \cap \epsilon_{uj}|s_u) &\\ = E_\alpha[P(\epsilon_{ui} \cap \epsilon_{uj}|\alpha, s_u)] &\\ = \int_0^\infty \Psi(\rho_{uij}, \phi_{ui}(\alpha), \phi_{uj}(\alpha)) \frac{\alpha}{\sigma^2} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) d\alpha. & \qquad (12) \end{aligned}$$

We can apply the KAT lower, stepwise lower, and greedy upper bounds [9] on (1) to obtain two lower bounds and one upper bound on P_s in terms of $P(\epsilon_{ui}|s_u), P(\epsilon_{ui} \cap \epsilon_{uj}|s_u)$

$$\Psi(\rho, a, b) = \begin{cases} \Psi(\rho, a, 0) + \Psi(-\rho, a, 0) - \Psi(-\rho, a, -b), & a \geq 0, b < 0 \\ \Psi(\rho, 0, b) + \Psi(-\rho, 0, b) - \Psi(-\rho, -a, b), & a < 0, b \geq 0 \\ 1 - \Psi(\rho, 0, -b) - \Psi(-\rho, 0, -b) & \\ \quad -\Psi(\rho, -a, 0) - \Psi(-\rho, -a, 0) + \Psi(\rho, -a, -b), & a < 0, b < 0 \end{cases} \qquad (11)$$

and $P(s_u)$. Note that (12) can be efficiently computed via the Gaussian quadrature method.

B. Bit-Error Rate (BER)

In many situations, the BER P_b is a more useful performance measure. Under the symbol MAP decoding criterion, P_b can be expressed as [9]

$$P_b = \sum_{u=1}^{M} P_b(u)P(s_u)$$

where

$$\begin{aligned} P_b(u) &= \frac{1}{\log_2 M} E(\sharp \text{ of bit errors}|s_u) \\ &= \frac{1}{\log_2 M}\sum_{m=1}^{M} d_H(w_m, w_u)A_{m|u} \\ A_{m|u} &= P(s_m \text{ is decoded}|s_u) \\ &= 1 - P\left(\bigcup_{i\neq m} \epsilon_{mi} \middle| s_u\right) \end{aligned}$$

where $u = 1, \ldots, M, w_m$ and w_u are the bit assignments for signals s_m and s_u, respectively, $d_H(w_m, w_u)$ is the Hamming distance between w_m and w_u, and ϵ_{mi} represents the event that symbol s_i has a higher metric than symbol s_m.

As in the case of the SER, $P(\epsilon_{mi}|\alpha, s_u)$ and $P(\epsilon_{mi} \cap \epsilon_{mj}|\alpha, s_u)$ can be expressed in terms of the $Q(\cdot)$ and $\Psi(\cdot)$ functions, respectively

$$\begin{aligned} &P(\epsilon_{mi}|\alpha, s_u) \\ &\quad = \Pr\{f(r|s_i,\alpha)P(s_i) \geq f(r|s_m,\alpha)P(s_m)|\alpha, s_u\} \\ &\quad = Q(\phi_{umi}(\alpha)) \qquad (13) \\ &P(\epsilon_{mi} \cap \epsilon_{mj}|\alpha, s_u) \\ &\quad = \Pr\{f(r|s_i,\alpha)P(s_i) \geq f(r|s_m,\alpha)P(s_m), \\ &\qquad f(r|s_j,\alpha)P(s_j) \geq f(r|s_m,\alpha)P(s_m)|\alpha, s_u\} \\ &\quad = \Psi(\rho_{mij}, \phi_{umi}(\alpha), \phi_{umj}(\alpha)) \qquad (14) \end{aligned}$$

where

$$\begin{aligned} \rho_{mij} &= \frac{\langle s_i - s_m, s_j - s_m\rangle}{\|s_i - s_m\| \cdot \|s_j - s_m\|} \qquad (15) \\ \phi_{umi}(\alpha) &= \frac{\sqrt{2N_0}\omega_{mi}}{2\alpha d_{mi}} + \frac{\alpha d_{ui}^2}{\sqrt{2N_0}d_{mi}} - \frac{\alpha d_{um}^2}{\sqrt{2N_0}d_{mi}}. \end{aligned}$$

$\Psi(\rho_{mij}, \cdot, \cdot)$ in (14) is given by (6) for $|\rho_{mij}| < 1$; otherwise, we have

$$\begin{aligned} \Psi(1, \phi_{umi}(\alpha), \phi_{umj}(\alpha)) &= Q(\max(\phi_{umi}(\alpha), \phi_{umj}(\alpha))) \\ \Psi(-1, \phi_{umi}(\alpha), \phi_{umj}(\alpha)) &= [Q(\phi_{umi}(\alpha)) - Q(-\phi_{umj}(\alpha))]^+. \end{aligned}$$

Therefore

$$P(\epsilon_{mi}|s_u) = E_\alpha[P(\epsilon_{mi}|\alpha, s_u)] = E_\alpha[Q(\phi_{umi}(\alpha))]. \qquad (16)$$

By integrating the RHS of (16) by parts, we can show (17) at the bottom of the page, where

$$\nu_{umi} = \sqrt{[\sigma^2(d_{ui}^2 - d_{um}^2)^2 + (2N_0 d_{mi}^2)]/\sigma^2}.$$

If $\phi_{umi}(\alpha)$ and $\phi_{umj}(\alpha)$ are nonnegative, we can use the result of [13] to write (6) as

$$\begin{aligned} &\Psi(\rho, \phi_{umi}(\alpha), \phi_{umj}(\alpha)) \\ &\quad = \frac{1}{2\pi}\int_0^{\vartheta(\phi_{umi}(\alpha)/\phi_{umj}(\alpha))} \exp\left[-\frac{\phi_{umi}(\alpha)^2}{2\sin^2\Phi}\right] d\Phi \\ &\qquad + \frac{1}{2\pi}\int_0^{\vartheta(\phi_{umj}(\alpha)/\phi_{umi}(\alpha))} \exp\left[-\frac{\phi_{umj}(\alpha)^2}{2\sin^2\Phi}\right] d\Phi. \qquad (18) \end{aligned}$$

Similarly, in the case where one or both of the second and the third arguments in the $\Psi(\cdot)$ function (6) are negative, the equalities given in (11) can be used. Therefore

$$\begin{aligned} &P(\epsilon_{mi} \cap \epsilon_{mj}|s_u) \\ &\quad = E_\alpha[P(\epsilon_{mi} \cap \epsilon_{mj}|\alpha, s_u)] \\ &\quad = \int_0^\infty \Psi(\rho_{mij}, \phi_{umi}(\alpha), \phi_{umj}(\alpha))\frac{\alpha}{\sigma^2} \\ &\qquad \times \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) d\alpha. \qquad (19) \end{aligned}$$

Applying the bounds of [9] to $P(\bigcup_{i\neq m} \epsilon_{mi}|s_u)$ yields two upper bounds and one lower bound on the BER P_b.

III. Numerical Results

We apply the KAT, stepwise, and greedy bounds to estimate P_s and P_b for 8, 16-PSK/16, 64-QAM signaling with Gray mapping over Rayleigh fading channels with $p = 0.5$ and 0.9. The results are shown in Figs. 1–4 in terms of the SNR E_b/N_0, where E_b is the energy per information bit. To verify the accuracy of the bounds, which were practical to compute, we also provide simulations obtained by averaging 1000 trials with 100 000 symbols each. For the results of the PSK constellations, we use the standard Gray code bit mapping (the binary reflected Gray code [4], [10]). For the QAM constellations, we use the Gray mappings shown in [15, Figs. 8 and 9] for 16-QAM and 64-QAM, respectively.

For uniform signaling ($p = 0.5$) with Gray mapped M-PSK and M-QAM, there exist exact or good approximations for P_s and P_b (e.g., [10], [11]). Note that in this case, MAP decoding is equivalent to maximum-likelihood decoding. Nevertheless, as shown in Figs. 1–3, our bounds (particularly the stepwise/greedy-based bounds) show excellent accuracy. For

$$P(\epsilon_{mi}|s_u) = \begin{cases} \frac{1}{2}\exp\left\{-\frac{\omega_{mi}}{2d_{mi}^2}\left[\left(d_{ui}^2 - d_{um}^2\right) + \nu_{umi}\right]\right\} \times \left(1 - \frac{\left(d_{ui}^2 - d_{um}^2\right)}{\nu_{umi}}\right), & \text{if } \omega_{mi} \geq 0 \\ 1 - \frac{1}{2}\exp\left\{\frac{\omega_{mi}}{2d_{mi}^2}\left[-\left(d_{ui}^2 - d_{um}^2\right) + \nu_{umi}\right]\right\} \times \left(1 + \frac{\left(d_{ui}^2 - d_{um}^2\right)}{\nu_{umi}}\right), & \text{if } \omega_{mi} < 0 \end{cases} \qquad (17)$$

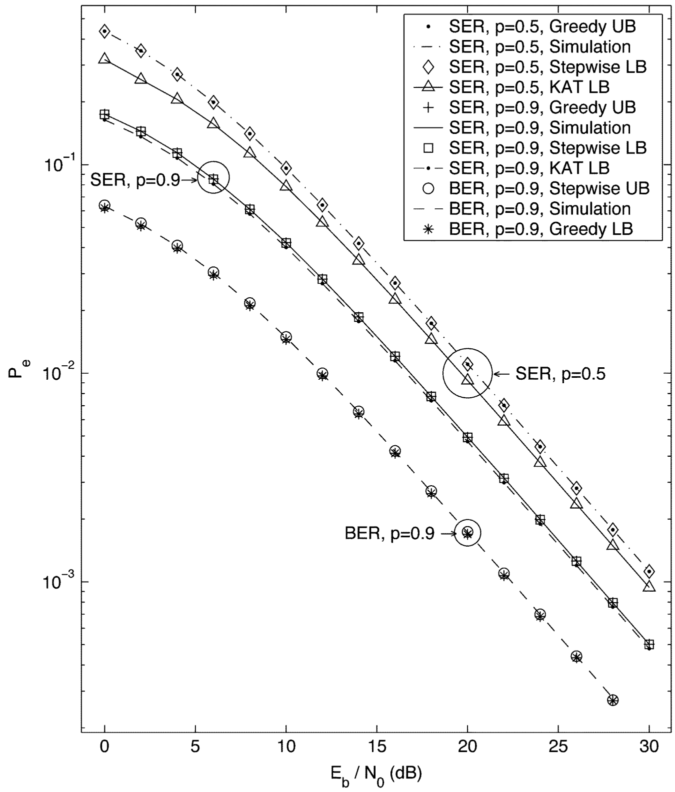

Fig. 1. SER P_s and BER P_b for 8-PSK with Gray mapping.

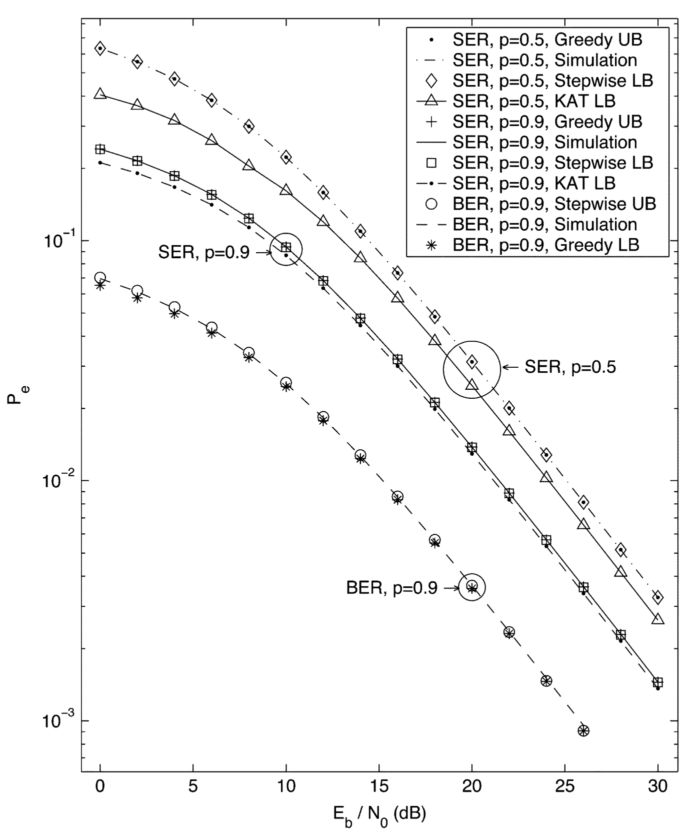

Fig. 2. SER P_s and BER P_b for 16-PSK with Gray mapping.

nonuniform signaling, exact or approximate expressions for P_s or P_b are not available (to the best of our knowledge); hence,

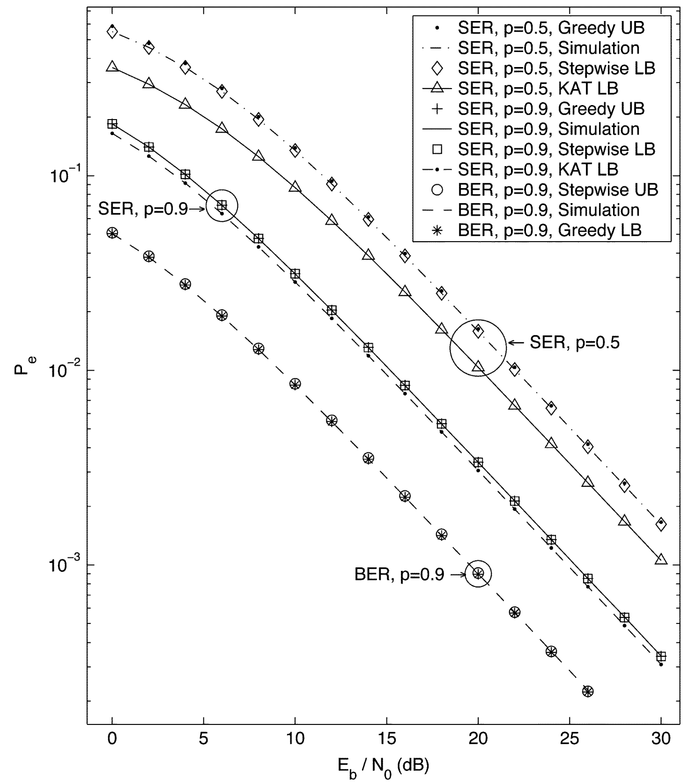

Fig. 3. SER P_s and BER P_b for 16-QAM with Gray mapping.

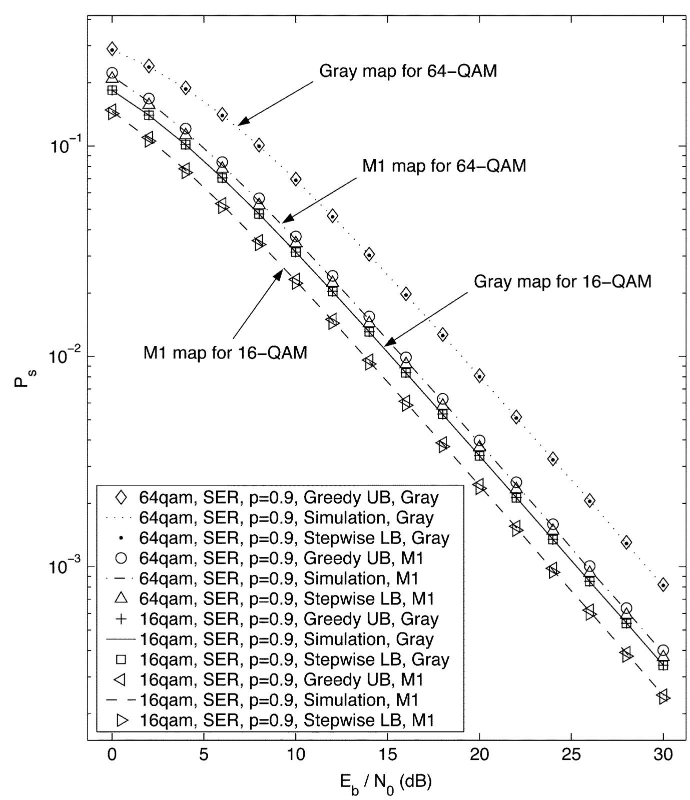

Fig. 4. SER P_s for 16, 64-QAM with Gray mapping and M1 mapping.

the need for accurate bounds is even more crucial. In Figs. 1–3, P_s and P_b performance curves with Gray mapping for $p = 0.9$ are also shown. Here also, the bounds provide an excellent estimate of the error probabilities over the entire range of

SNRs. The stepwise and the greedy bounds are particularly impressive, as they agree with each other and the simulations even during very severe channel conditions. The KAT lower bound is weaker than the stepwise bound; but when $p = 0.9$, it is noticeably tighter than when $p = 0.5$.

We also estimate P_s for systems with $p = 0.9$ using the heuristic M_1 map constructed for 16-QAM (see [15, Fig. 8]) and 64-QAM (see [15, Fig. 9]) constellations. As shown in Fig. 4, the bounds are very accurate, and the M_1 map considerably outperforms the Gray map. For $5 \times 10^{-4} \leq P_s \leq 10^{-2}$, gains up to 1.5 and 3.3 dB in E_b/N_0 are achieved for 16-QAM and 64-QAM, respectively.

Finally, equipped with the simple analytical closed-form expression derived in (9) for the conditional individual error-event probabilities (or pairwise error probabilities) $P(\epsilon_{ui}|s_u)$, we study the merits of the mapping M_1 for the Rayleigh fading channel and strongly nonuniform signaling. Specifically, we search for the map that minimizes the union upper bound on the SER, given by $\sum_{u=1}^{M} P(s_u) \sum_{i \neq u} P(\epsilon_{ui}|s_u)$, over the sets of maps with minimum average symbol energy. To satisfy the smallest average energy constraint, we perform our search according to the design criteria developed in [15]. We then compare the performance of the M_1 mapping with that of the optimal map yielded by the search. Typical results, obtained for the case of $p = 0.9$ and 16-QAM, show that the best map is very close to that of the M_1 mapping. This indicates that the M_1 mapping is indeed a very good choice for the Rayleigh fading channel when used with nonuniform signaling. Note also that the search is computationally efficient by virtue of the simple expression for $P(\epsilon_{ui}|s_u)$.

IV. Conclusion

In this letter, we derive the KAT, stepwise, and greedy bounds for the error analysis of Rayleigh fading channels with nonuniform signaling and MAP decoding. The stepwise and greedy bounds show excellent accuracy for both BER and SER in conjunction with the Gray and M_1 [15] mappings over the entire range of SNR values, even those corresponding to very severe channel conditions. As a byproduct of our results, we use the closed-form error expressions given by (9) to validate the choice of the M_1 mapping for the Rayleigh fading channel when the source is strongly biased. This is achieved by observing that it performs nearly identically to the best map with smallest average symbol energy which minimizes the SER union upper bound. It is important to mention that the bounds are general, in the sense that they are independent of the properties of the signal mapping and the geometry of the signaling constellation. Future work includes the study of these bounds for multiantenna space–time coded communication systems. Preliminary results in this direction are reported in [2].

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