On Bounding the Union Probability Using Partial Weighted Information[☆]

Jun Yang[a], Fady Alajaji[b], Glen Takahara[b]

[a] *Department of Statistical Sciences, University of Toronto, Canada*
[b] *Department of Mathematics and Statistics, Queen's University, Canada*

Abstract

Lower bounds on the finite union probability are established in terms of the individual event probabilities and a weighted sum of the pairwise event probabilities. The lower bounds have at most pseudo-polynomial computational complexity and generalize recent analytical bounds.

Keywords: Probability of a finite union of events, lower and upper bounds, optimal bounds, linear programming.

1. Introduction

Lower and upper bounds on the union probability $P\left(\bigcup_{i=1}^{N} A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s have been actively investigated in the recent past. Optimal bounds can be obtained numerically by solving linear programming (LP) problems with 2^N variables (for instance, see [1, 2]). Since the number of variables is exponential in the number of events, N, some suboptimal but numerically efficient bounds have been proposed, such as the algorithmic Bonferroni-type bounds in [3, 4].

Among the established analytical bounds is the Kuai-Alajaji-Takahara lower bound (for convenience, hereafter referred to as the KAT bound) [5] that was

☆This work was supported in part by NSERC of Canada.

Email addresses: `jun@utstat.toronto.edu` (Jun Yang), `fady@mast.queensu.ca` (Fady Alajaji), `takahara@mast.queensu.ca` (Glen Takahara)

shown to be better than the Dawson-Sankoff (DS) [6] and the D. de Caen (DC) bounds [7]. Noting that the KAT bound is expressed in terms of $\{P(A_i)\}$ and only the *sums* of the pairwise event probabilities, i.e., $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, in order to fully exploit all pairwise event probabilities, it is observed in [8, 9, 10] that the analytical bounds can be further improved algorithmically by optimizing over subsets. Furthermore, in [1], the KAT bound is extended by using additional partial information such as the sums of joint probabilities of three events, i.e., $\{\sum_{j,l} P(A_i \cap A_j \cap A_l), i = 1, \ldots, N\}$. Recently, using the same partial information as the KAT bound, i.e., $\{P(A_i)\}$ and $\{\sum_{j:j\neq i} P(A_i \cap A_j)\}$, the optimal lower/upper bound as well as a new analytical bound which is sharper than the KAT bound were developed by Yang-Alajaji-Takahara in [11, 12] (these two bounds are respectively referred to as the YAT-I and YAT-II bounds).

In this work, we extend the existing analytical lower bounds, the KAT and YAT-II bounds, and establish two new classes of lower bounds on $P\left(\bigcup_{i=1}^{N} A_i\right)$ using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$ for a given weight or parameter vector $\boldsymbol{c} = (c_1, \ldots, c_N)^T$. These lower bounds are shown to have at most pseudo-polynomial computational complexity and to be sharper in certain cases than the Gallot-Kounias (GK) [13, 14] and Prékopa-Gao (PG) bounds [1] even though the latter bounds employ more information on the events joint probabilities.

More specifically, we first propose a novel expression for the union probability given a weight vector $\boldsymbol{c}$. Using the Cauchy-Schwarz inequality, several existing bounds, such as the bound in [15], and the DC and GK bounds, can be directly derived from this new expression. Next, we derive two new classes of lower bounds as functions of the weight vector $\boldsymbol{c}$ by solving linear programming problems. The KAT and YAT-II analytical bounds are shown to be special cases of the new classes of lower bounds. Furthermore, it is noted that the proposed lower bounds can be sharper than the GK bound under some conditions.

We emphasize that our bounds can be applied to any general estimation problem involving the probability of a finite union of events. In particular, they can be applied to effectively estimate and analyze the error performance of communication systems (e.g., see [12, 3, 8, 15, 16, 17]). Such bounds are also pertinently useful in the analysis of asymptotic problems such as the Borel-Cantelli Lemma and its generalization [18, 19, 20, 21]. Finally, we note that the proposed bounds provide useful tools for chance-constrained stochastic programs (e.g., see [22, 23]) in operations research. More specifically, using partial information of uncertainty, the proposed bounds on the union probability can be applied to formulate tractable conservative approximations of chance-constrained stochastic problems, which can be solved efficiently and produce feasible solutions for the original problems (see, for instance, [24, 25, 26]). An example of such application is the work in [27] on the probabilistic set covering problem with correlations, where the KAT bound is used for the case where only partial information on the correlation is available.

The outline of this letter is as follows. In Section 2, we propose a new expression of the union probability using weight vector $\boldsymbol{c}$ such that many existing bounds can be directly derived from this expression. In Section 3, we develop two new classes of lower bounds as functions of the weight vector $\boldsymbol{c}$ and discuss their connection with existing bounds, including the KAT, YAT-II and GK bounds. As by-products of the new lower bounds, two new classes of upper bounds are also obtained. Finally, in Section 4, we compare via numerical examples existing lower bounds with the proposed bounds under different choices of weight vectors.

2. A New Expression of the Union Probability

For simplicity, and without loss of generality, we assume that the events $\{A_1, \ldots, A_N\}$ are in a finite probability space $(\Omega, \mathscr{F}, P)$, where N is a fixed positive integer. Let $\mathscr{B}$ denote the collection of all non-empty subsets of $\{1, 2, \ldots, N\}$. Given $B \in \mathscr{B}$, we let ω_B denote the atom in $\cup_{i=1}^{N} A_i$ such that for all $i = 1, \ldots, N$, $\omega_B \in A_i$ if $i \in B$ and $\omega_B \notin A_i$ if $i \notin B$ (note that some of these "atoms" may be the empty set). For ease of notation, for a singleton $\omega \in \Omega$, we denote $P(\{\omega\})$ by $p(\omega)$ and $P(\omega_B)$ by p_B. Since $\{\omega_B : i \in B\}$ is the collection of all the atoms in A_i, we have $P(A_i) = \sum_{\omega \in A_i} p(\omega) = \sum_{B \in \mathscr{B}: i \in B} p_B$, and

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B\in\mathscr{B}} p_B. \tag{1}$$

Suppose there are N functions $f_i(B), i = 1, \ldots, N$ such that $\sum_{i=1}^{N} f_i(B) = 1$ for any $B \in \mathscr{B}$. If we further assume that $f_i(B) = 0$ if $i \notin B$, we can write

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B\in\mathscr{B}} \left(\sum_{i=1}^{N} f_i(B)\right) p_B = \sum_{i=1}^{N} \sum_{B\in\mathscr{B}:i\in B} f_i(B) p_B. \tag{2}$$

Note that if we define the degree of a subset $A \subset \Omega$, $\deg(A)$, to be the number of A_i's that contain A, then by the definition of ω_B, we have $\deg(\omega_B) = |B|$. Therefore,

$$f_i(B) = \begin{cases} \frac{1}{|B|} = \frac{1}{\deg(\omega_B)} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \tag{3}$$

satisfies $\sum_{i=1}^{N} f_i(B) = 1$ and (2) becomes

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \sum_{B\in\mathscr{B}:i\in B} \frac{p_B}{\deg(\omega_B)} = \sum_{i=1}^{N} \sum_{\omega\in A_i} \frac{p(\omega)}{\deg(\omega)}. \tag{4}$$

Note that many of the existing bounds, such as the DC bound, the KAT bound and the recent bounds in [11] and [12], are based on (4).

In the following lemma, we propose a generalized expression of (4).

Lemma 1. *Suppose $\{\omega_B, B \in \mathscr{B}\}$ are all the $2^N - 1$ atoms in $\bigcup_i A_i$. If $\boldsymbol{c} = (c_1, \ldots, c_N)^T \in \mathbb{R}^N$ satisfies*

$$\sum_{k\in B} c_k \neq 0, \quad \textit{for all} \quad B \in \mathscr{B} \tag{5}$$

then we have

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \sum_{B\in\mathscr{B}:i\in B} \frac{c_i p_B}{\sum_{k\in B} c_k} = \sum_{i=1}^{N} \sum_{\omega\in A_i} \frac{c_i p(\omega)}{\sum_{\{k:\omega\in A_k\}} c_k}. \tag{6}$$

Proof. If we define

$$f_i(B) = \begin{cases} \frac{c_i}{\sum_{k \in B} c_k} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases}$$

where the parameter vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)^T$ satisfies $\sum_{k \in B} c_k \neq 0$ for all $B \in \mathscr{B}$ (therefore $c_i \neq 0, i = 1, \ldots, N$), then $\sum_i f_i(\omega) = 1$ holds and we can get (6) from (2). ■

Note that (6) holds for any $\boldsymbol{c}$ that satisfies (5) and is clearly a generalized expression of (4).

Remark 1. *Both the Cohen-Merhav [15] and Gallot-Kounias [13] bounds can be derived from this new expression of the union probability in Lemma 1 using the Cauchy-Schwarz inequality. We refer to [28, Sections 1 and 2] for further details.*

3. New Bounds using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$

Due to space limitations, we only present the results when $\boldsymbol{c} \in \mathbb{R}_+^N$. Results regarding more general $\boldsymbol{c}$ are available in the accompanying supplementary material [28, Section 3].

3.1. New Class of Lower Bounds when $\boldsymbol{c} \in \mathbb{R}_+^N$ satisfies (5)

Theorem 1. *For any given $\boldsymbol{c} \in \mathbb{R}_+^N$ that satisfies (5), a new lower bound on the union probability is given by*

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \ell_i(\boldsymbol{c}) =: \ell_{NEW\text{-}I}(\boldsymbol{c}), \tag{7}$$

where

$$\ell_i(\boldsymbol{c}) = P(A_i)\left(\frac{c_i}{\sum_{k \in B_1^{(i)}} c_k} + \frac{c_i}{\sum_{k \in B_2^{(i)}} c_k} - \frac{c_i \sum_k c_k P(A_i \cap A_k)}{P(A_i)\left(\sum_{k \in B_1^{(i)}} c_k\right)\left(\sum_{k \in B_2^{(i)}} c_k\right)}\right), \tag{8}$$

where $B_1^{(i)}$ and $B_2^{(i)}$ are subsets of $\{1,\ldots,N\}$ that satisfy

$$\begin{aligned} B_1^{(i)} &= \arg\max_{\{B: i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad s.t. \quad \frac{\sum_{k\in B} c_k}{c_i} \le \frac{\sum_k c_k P(A_i\cap A_k)}{c_i P(A_i)}, \\ B_2^{(i)} &= \arg\min_{\{B: i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad s.t. \quad \frac{\sum_{k\in B} c_k}{c_i} \ge \frac{\sum_k c_k P(A_i\cap A_k)}{c_i P(A_i)}. \end{aligned} \tag{9}$$

Proof. The proof is given in [28, Section 3]. ■

Remark 2 (The new bound $\ell_{\mathbf{NEW\text{-}I}}(\boldsymbol{c})$ v.s. the GK bound $\ell_{\mathbf{GK}}$). *For any $\boldsymbol{c}\in\mathbb{R}_+^N$, we have these relations between different lower bounds:*

$$\begin{aligned} \ell_{NEW\text{-}I}(\boldsymbol{c}) &\ge \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i\cap A_k)} \\ &\ge \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i\sum_k c_i c_k P(A_i\cap A_k)} \ge \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i\sum_j c_i^2 P(A_i\cap A_j)}. \end{aligned} \tag{10}$$

Therefore, if the optimal weight vector obtained by the GK bound (see see [28, Eq. (21)] and [29]), denoted by $\tilde{\mathbf{c}}$, satisfies $\tilde{\mathbf{c}}\in\mathbb{R}_+^N$, then $\ell_{NEW\text{-}I}(\tilde{\mathbf{c}}) \ge \ell_{GK}$.

Remark 3 (The new bound $\ell_{\mathbf{NEW\text{-}I}}(\boldsymbol{c})$ v.s. the KAT bound $\ell_{\mathbf{KAT}}$). *One can easily verify that $\ell_{NEW\text{-}I}(\kappa\mathbf{1}) = \ell_{KAT}$, where $\mathbf{1}$ is the all-one vector of size N and κ is any non-zero constant.*

Lemma 2. *When $\mathbf{c}\in\mathbb{R}_+^N$, the lower bound $\ell_{NEW\text{-}I}(\mathbf{c})$ can be computed in pseudo-polynomial time, and can be arbitrarily closely approximated by an algorithm running in polynomial time.*

Proof. See [28, Section 4]. ■

Corollary 1. *(New class of upper bounds $\hbar_{NEW\text{-}I}(\mathbf{c})$): We can derive an upper bound for any given $\mathbf{c}\in\mathbb{R}_+^N$ by*

$$\begin{aligned} P\left(\bigcup_i A_i\right) \le \left(\frac{1}{\min_k c_k} + \frac{1}{\sum_k c_k}\right)\sum_i c_i P(A_i) \\ - \frac{1}{(\min_k c_k)\sum_k c_k}\sum_i\sum_k c_i c_k P(A_i\cap A_k) =: \hbar_{NEW\text{-}I}(\boldsymbol{c}). \end{aligned} \tag{11}$$

The proof is given in [28, Section 5]. According to the results from randomly generated $\boldsymbol{c}$, it is conjectured the optimal upper bound in this class is achieved at $\boldsymbol{c} = \kappa\mathbf{1}$ where κ is any non-zero constant.

3.2. New Class of Lower Bounds when $\boldsymbol{c} \in \mathbb{R}_+^N$

We only consider $\boldsymbol{c} \in \mathbb{R}_+^N$ in this subsection. A new class of lower bounds, $\ell_{\text{NEW-II}}$, is given in the following theorem.

Theorem 2. *Defining $\mathscr{B}^- = \mathscr{B} \setminus \{1, \ldots, N\}$, $\tilde{\gamma}_i := \sum_k c_k P(A_i \cap A_k)$, $\tilde{\alpha}_i := P(A_i)$ and*

$$\tilde{\delta} := \max_i \left[\frac{\tilde{\gamma}_i - \left(\sum_k c_k - \min_k c_k\right) \tilde{\alpha}_i}{\min_k c_k} \right]^+, \tag{12}$$

where $\boldsymbol{c} \in \mathbb{R}_+^N$, another class of lower bounds is given by

$$P\left(\bigcup_{i=1}^N A_i\right) \geq \tilde{\delta} + \sum_{i=1}^N \ell_i'(\boldsymbol{c}, \tilde{\delta}) =: \ell_{NEW\text{-}II}(\boldsymbol{c}), \tag{13}$$

where

$$\ell_i'(\boldsymbol{c}, x) = [P(A_i) - x] \cdot \left(\frac{c_i}{\sum_{k \in B_1^{(i)}} c_k} + \frac{c_i}{\sum_{k \in B_2^{(i)}} c_k} - \frac{c_i \sum_k c_k \left[P(A_i \cap A_k) - x\right]}{\left[P(A_i) - x\right] \left(\sum_{k \in B_1^{(i)}} c_k\right) \left(\sum_{k \in B_2^{(i)}} c_k\right)} \right), \tag{14}$$

and

$$\begin{aligned} B_1^{(i)} &= \arg \max_{\{B \in \mathscr{B}^- : i \in B\}} \frac{\sum_{k \in B} c_k}{c_i} \quad s.t. \quad \frac{\sum_{k \in B} c_k}{c_i} \leq \frac{\sum_k c_k \left[P(A_i \cap A_k) - x\right]}{c_i \left[P(A_i) - x\right]}, \\ B_2^{(i)} &= \arg \min_{\{B \in \mathscr{B}^- : i \in B\}} \frac{\sum_{k \in B} c_k}{c_i} \quad s.t. \quad \frac{\sum_{k \in B} c_k}{c_i} \geq \frac{\sum_k c_k \left[P(A_i \cap A_k) - x\right]}{c_i \left[P(A_i) - x\right]}. \end{aligned} \tag{15}$$

Proof. Let $x = p_{\{1,2,\ldots,N\}}$ and consider $\sum_i \ell_i'(\boldsymbol{c}, x) + x$ as a new lower bound where where $\ell_i'(\boldsymbol{c}, x)$ equals to the objective value of the problem

$$\begin{aligned}
&\min_{\{p_B : i\in B, B\in\mathscr{B}^-\}} \sum_{B:i\in B, B\in\mathscr{B}^-} \frac{c_i p_B}{\sum_{k\in B} c_k} \\
&\text{s.t.} \quad \sum_{B:i\in B, B\in\mathscr{B}^-} p_B = P(A_i) - x, \\
&\qquad \sum_{B:i\in B, B\in\mathscr{B}^-} \left(\frac{\sum_{k\in B} c_k}{c_i}\right) p_B = \frac{1}{c_i}\sum_k c_k \left[P(A_i \cap A_k) - x\right], \\
&\qquad p_B \geq 0, \quad \text{for all} \quad B \in \mathscr{B}^- \quad \text{such that} \quad i \in B.
\end{aligned} \tag{16}$$

The solution of (16) exists if and only if

$$\min_k c_k \leq \frac{\tilde{\gamma}_i - (\sum_k c_k)x}{\tilde{\alpha}_i - x} \leq \sum_k c_k - \min_k c_k. \tag{17}$$

Therefore, the new lower bound can be written as

$$\begin{aligned}
&\min_x \left[x + \sum_{i=1}^N \ell_i'(\boldsymbol{c}, x)\right] \\
&\text{s.t.} \quad \left[\frac{\tilde{\gamma}_i - (\sum_k c_k - \min_k c_k)\,\tilde{\alpha}_i}{\min_k c_k}\right]^+ \leq x \leq \frac{\tilde{\gamma}_i - (\min_k c_k)\tilde{\alpha}_i}{\sum_k c_k - \min_k c_k}, \forall i.
\end{aligned} \tag{18}$$

We can prove that the objective function of (18) is non-decreasing with x. Therefore, defining $\tilde{\delta}$ as in (12), the new lower bound can be written as (13) where $\ell_i'(\boldsymbol{c}, \tilde{\delta})$ can be obtained by solving (16), which is given in (14). We refer to [28, Section 6] for more details for the proof. ■

Remark 4 ($\ell_{\mathbf{NEW\text{-}II}}(\boldsymbol{c})$ v.s. $\ell_{\mathbf{NEW\text{-}I}}(\boldsymbol{c})$). *If $\boldsymbol{c} \in \mathbb{R}_+^N$, $\ell_{NEW\text{-}I}(\boldsymbol{c})$ is the solution of a relaxed problem to the problem for obtaining $\ell_{NEW\text{-}II}(\boldsymbol{c})$; thus*

$$\ell_{NEW\text{-}II}(\boldsymbol{c}) \geq \ell_{NEW\text{-}I}(\boldsymbol{c}).$$

Also, since $\mathbf{c} \in \mathbb{R}_+^N$, the solution of (15) can be computed in pseudo-polynomial time and has a polynomial-time approximation algorithm.

Remark 5 (The new bound $\ell_{\mathbf{NEW\text{-}II}}(\boldsymbol{c})$ v.s. the YAT-II bound $\ell_{\mathbf{YAT\text{-}II}}$). *One can easily verify that $\ell_{NEW\text{-}II}(\kappa\mathbf{1}) = \ell_{YAT\text{-}II}$, where $\mathbf{1}$ is the all-one vector of size N and κ is any non-zero constant.*

Corollary 2. *(Improved class of upper bounds $\hbar_{NEW\text{-}II}(\boldsymbol{c})$): We can improve the upper bound $\hbar_{NEW\text{-}I}(\boldsymbol{c})$ in (11) by*

$$\begin{aligned} P\left(\bigcup_i A_i\right) \leq \min_i &\left\{\frac{\sum_k c_k P(A_i \cap A_k) - (\min_k c_k) P(A_i)}{\sum_k c_k - \min_k c_k}\right\} \\ &+ \left(\frac{1}{\min_k c_k} + \frac{1}{\sum_k c_k - \min_k c_k}\right) \sum_i c_i P(A_i) \\ &- \frac{1}{(\min_k c_k)(\sum_k c_k - \min_k c_k)} \sum_i \sum_k c_i c_k P(A_i \cap A_k) =: \hbar_{NEW\text{-}II}(\boldsymbol{c}). \end{aligned} \tag{19}$$

Note that the upper bound $\hbar_{NEW\text{-}II}(\boldsymbol{c})$ in (19) is always sharper than $\hbar_{NEW\text{-}I}$ in (11). The proof is given in [28, Section 7]. According to numerical examples using randomly generated $\mathbf{c}$, it is conjectured the optimal upper bound in this class is achieved at $\mathbf{c} = \kappa\mathbf{1}$, where κ is any non-negative constant.

4. Numerical Examples

The same eight systems as in [11] are used in this section. For comparison, we include bounds that utilize $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j), i = 1, \ldots, N\}$, such as ℓ_{KAT}, $\ell_{\mathrm{YAT\text{-}II}}$ and the optimal lower bound $\ell_{\mathrm{YAT\text{-}I}}$ in this class. Furthermore, we included the GK bound ℓ_{GK} which fully exploits $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$

and the PG bound [1], denoted as ℓ_{PG}, which extends the KAT bound by using $\{P(A_i)\}$, $\{\sum_j P(A_i \cap A_j)\}$ and $\{\sum_{j,l} P(A_i \cap A_j \cap A_l)\}$.

In the numerical examples, $\tilde{\boldsymbol{c}}$ is obtained by the GK bound (see [28, Eq. (21)] or [29]); the elements of $\tilde{\boldsymbol{c}}^+$ are given by $\{\tilde{c}_i^+ = \max(\tilde{c}_i, \epsilon), i = 1, \ldots, N\}$ where $\epsilon > 0$ is small enough so that if $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$ then $\tilde{\boldsymbol{c}}^+ = \tilde{\boldsymbol{c}}$.

We present $\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$, $\ell_{\text{NEW-II}}(\tilde{\boldsymbol{c}}^+)$ and $\max_\kappa \ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}} + \kappa \mathbf{1})$ in Table 1. In three examples (Systems II, III and VIII), $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$; therefore $\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}) = \ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$. The lower bound $\max_\kappa \ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}} + \kappa \mathbf{1})$ is calculated by searching κ from -1 to 1 with a fixed step length 0.005 (so that 401 points are used in total). We also randomly generated $100,000$ samples of $\boldsymbol{c} \in \mathbb{R}_+^N$ to compute $\ell_{\text{NEW-I}}(\boldsymbol{c})$ and $\ell_{\text{NEW-II}}(\boldsymbol{c})$ and the largest bounds were selected and denoted as $\ell_{\text{NEW-I}}(\boldsymbol{c}_{\text{Rand}}^+)$ and $\ell_{\text{NEW-II}}(\boldsymbol{c}_{\text{Rand}}^+)$.

From the results, we note that $\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$ is sharper than ℓ_{GK} in most of the examples except for System VI. The line search $\max_\kappa \ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}} + \kappa \mathbf{1})$ is sharper than $\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$ in most of the examples except for Systems I and V. Since $\tilde{\boldsymbol{c}}^+ \in \mathbb{R}_+^N$, the class of lower bounds $\ell_{\text{NEW-II}}(\tilde{\boldsymbol{c}}^+)$ is at least as good as $\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$, as observed in Remark 4 (in the examples shown in the table, both bounds give identical results). Furthermore, the PG bound which uses sums of joint probabilities of three events, may be even poorer (e.g., see Systems I and VI) than the numerical bound $\ell_{\text{YAT-I}}$ which utilizes less information but is optimal in the class of lower bounds using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. It is also weaker than the proposed lower bounds in several cases (see Systems I-IV).

In Table 2, we compared $\ell_{\text{NEW-I}}(\boldsymbol{c})$ and $\ell_{\text{NEW-II}}(\boldsymbol{c})$ with randomly generated $\boldsymbol{c} \in \mathbb{R}_+^N$. We remark that in System VI, the maximum $\ell_{\text{NEW-II}}(\boldsymbol{c})$ is 0.3203 which is sharper than the maximum $\ell_{\text{NEW-I}}(\boldsymbol{c})$ which is 0.3022. Also, the percentage that $\ell_{\text{NEW-II}}(\boldsymbol{c})$ is strictly larger than $\ell_{\text{NEW-I}}(\boldsymbol{c})$ and the averages of $\frac{\ell_{\text{NEW-II}}(\boldsymbol{c})}{\ell_{\text{NEW-I}}(\boldsymbol{c})}$ are shown in Table 2.

We close by noting that the new general lower and upper bounds established in this work are applicable to the analytical or numerical study of any statistical problem involving the probability of a finite union of events.

References

[1] A. Prékopa, L. Gao, Bounding the probability of the union of events by aggregation and disaggregation in linear programs, Discrete Applied Mathematics 145 (3) (2005) 444–454.

[2] P. Veneziani, Lower bounds of degree 2 for the probability of the union of N events via linear programming, unpublished (June 2007).

[3] H. Kuai, F. Alajaji, G. Takahara, Tight error bounds for nonuniform signaling over AWGN channels, IEEE Transactions on Information Theory 46 (7) (2000) 2712–2718.

[4] F. Behnamfar, F. Alajaji, T. Linder, Tight error bounds for space-time orthogonal block codes under slow Rayleigh flat fading, IEEE Transactions on Communications 53 (6) (2005) 952–956.

[5] H. Kuai, F. Alajaji, G. Takahara, A lower bound on the probability of a finite union of events, Discrete Mathematics 215 (1-3) (2000) 147–158.

[6] D. A. Dawson, D. Sankoff, An inequality for probabilities, Proceedings of the American Mathematical Society 18 (3) (1967) 504–507.

[7] D. De Caen, A lower bound on the probability of a union, Discrete Mathematics 169 (1) (1997) 217–220.

[8] F. Behnamfar, F. Alajaji, T. Linder, An efficient algorithmic lower bound for the error rate of linear block codes, IEEE Transactions on Communications 55 (6) (2007) 1093–1098.

[9] F. M. Hoppe, Improving probability bounds by optimization over subsets, Discrete Mathematics 306 (5) (2006) 526–530.

[10] F. M. Hoppe, The effect of redundancy on probability bounds, Discrete Mathematics 309 (1) (2009) 123–127.

[11] J. Yang, F. Alajaji, G. Takahara, Lower Bounds on the Probability of a Finite Union of Events, ArXiv e-prints, arXiv:1401.5543.

[12] J. Yang, F. Alajaji, G. Takahara, New bounds on the probability of a finite union of events, in: 2014 IEEE International Symposium on Information Theory (ISIT), 2014, pp. 1271–1275. doi:10.1109/ISIT.2014.6875037.

[13] S. Gallot, A bound for the maximum of a number of random variables, Journal of Applied Probability 3 (2) (1966) 556–558.

[14] E. G. Kounias, Bounds for the probability of a union, with applications, The Annals of Mathematical Statistics 39 (6) (1968) 2154–2158.

[15] A. Cohen, N. Merhav, Lower bounds on the error probability of block codes based on improvements on de Caen's inequality, IEEE Transactions on Information Theory 50 (2) (2004) 290–310.

[16] G. Seguin, A lower bound on the error probability for signals in white Gaussian noise, IEEE Transactions on Information Theory 44 (7) (1998) 3168–3175.

[17] Z. Mao, J. Cheng, J. Shen, A new lower bound on error probability for nonuniform signals over AWGN channels, in: Wireless Communications and Networking Conference (WCNC), IEEE, 2013, pp. 3005–3009.

[18] P. Erdős, A. Rényi, On Cantor's series with convergent $\sum 1/q_n$, Ann. Univ. Sci. Budapest. Eőtvős Sect. Math. 2 (1959) 93–109.

[19] C. Feng, L. Li, J. Shen, On the Borel-Cantelli lemma and its generalization, Comptes Rendus Mathematique I (347) (2009) 1313–1316.

[20] A. N. Frolov, Bounds for probabilities of unions of events and the Borel–Cantelli lemma, Statistics & Probability Letters 82 (12) (2012) 2189–2197.

[21] C. Feng, L. Li, On the Móri-Székely conjectures for the Borel-Cantelli lemma, Studia Scientiarum Mathematicarum Hungarica 50 (2) (2013) 280–285.

[22] A. Prékopa, Stochastic programming, Kluwer Academic Publishers Group, Dordrecht, 1995.

[23] A. Shapiro, D. Dentcheva, A. Ruszczyński, Lectures on stochastic programming: modeling and theory, Vol. 16, SIAM, 2014.

[24] J. Pintér, Deterministic approximations of probability inequalities, Zeitschrift für Operations Research 33 (4) (1989) 219–239.

[25] A. Nemirovski, A. Shapiro, Convex approximations of chance constrained programs, SIAM Journal on Optimization 17 (4) (2006) 969–996.

[26] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robust optimization, Princeton University Press, 2009.

[27] S. Ahmed, D. J. Papageorgiou, Probabilistic set covering with correlations, Operations Research 61 (2) (2013) 438–452.

[28] J. Yang, F. Alajaji, G. Takahara, On bounding the union probability using partial weighted information – supplementary material, Statistics and Probability Letters.

[29] C. Feng, L. Li, J. Shen, Some inequalities in functional analysis, combinatorics, and probability theory, The Electronic Journal of Combinatorics 17 (R58) (2010) 1.

Table 1: Comparison of lower bounds (* indicates $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$ and a bold number indicates the best results among all tested bounds.)

System	I	II	III	IV	V	VI	VII	VIII
N	6	6	6	7	3	4	4	4
$P\left(\bigcup_{i=1}^N A_i\right)$	0.7890	0.6740	0.7890	0.9687	0.3900	0.3252	0.5346	0.5854
ℓ_{KAT}	0.7247	0.6227	0.7222	0.8909	0.3833	0.2769	0.4434	0.5412
ℓ_{GK}	0.7601	0.6510	0.7508	0.9231	0.3813	0.2972	0.4750	0.5390
ℓ_{PG}	0.7443	0.6434	0.7556	0.9148	**0.3900**	0.3240	**0.5281**	**0.5726**
$\ell_{\text{YAT-II}}$	0.7247	0.6227	0.7222	0.8909	**0.3900**	0.3205	0.4562	0.5464
$\ell_{\text{YAT-I}}$	0.7487	0.6398	0.7427	0.9044	**0.3900**	**0.3252**	0.5090	0.5531
$\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+)$	0.7638	0.6517*	0.7512*	0.9231	**0.3900**	0.2951	0.4905	0.5412*
$\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}} + \kappa\mathbf{1})$	0.7577	0.6539	0.7557	0.9235	0.3899	0.2993	0.4949	0.5412
$\ell_{\text{NEW-I}}(\tilde{\boldsymbol{c}}^+_{\text{Rand}})$	**0.7783**	**0.6633**	**0.7810**	**0.9501**	**0.3900**	0.3022	0.4992	0.5666
$\ell_{\text{NEW-II}}(\tilde{\boldsymbol{c}}^+)$	0.7638	0.6517	0.7512	0.9231	**0.3900**	0.2951	0.4905	0.5412
$\ell_{\text{NEW-II}}(\tilde{\boldsymbol{c}}^+_{\text{Rand}})$	**0.7783**	**0.6633**	**0.7810**	**0.9501**	**0.3900**	0.3203	0.4992	0.5666

Table 2: Comparison of $\ell_{\text{NEW-I}}(\boldsymbol{c})$ and $\ell_{\text{NEW-II}}(\boldsymbol{c})$ with randomly generated $\boldsymbol{c} \in \mathbb{R}_+^N$ (a bold number indicates $\max \ell_{\text{NEW-II}}(\boldsymbol{c}) > \max \ell_{\text{NEW-I}}(\boldsymbol{c})$.)

System	V	VI	VII	VIII
N	3	4	4	4
$P\left(\bigcup_{i=1}^N A_i\right)$	0.3900	0.3252	0.5346	0.5854
$\max \ell_{\text{NEW-I}}(\boldsymbol{c})$	0.3900	0.3022	0.4992	0.5666
$\max \ell_{\text{NEW-II}}(\boldsymbol{c})$	0.3900	**0.3203**	0.4992	0.5666
Average $\frac{\ell_{\text{NEW-II}}(\boldsymbol{c})}{\ell_{\text{NEW-I}}(\boldsymbol{c})}$	1.0011	1.065	1.0006	1.0000
Percentage $\ell_{\text{NEW-II}}(\boldsymbol{c}) > \ell_{\text{NEW-I}}(\boldsymbol{c})$	7.82 %	69.6%	3.87%	0.54 %