Polynomial-like Behaviour of the Faithful Dimension of p-Groups

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The *faithful dimension* of a finite group G over $\mathbb{C}$, denoted by $m_{\text{faithful}}(G)$, is defined to be the smallest integer m such that G can be embedded in $GL_m(\mathbb{C})$. We are interested in computing the faithful dimension of a p-group of the form $\mathscr{G}_q := \exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} \mathbb{F}_q)$, where $q = p^f$ and $\mathfrak{f}_{n,c}$ is the free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators.

In 2019, Bardestani et al. expressed the faithful dimension of $\mathscr{G}_q$ as the solution to a rank minimization problem by applying Kirillov's orbit method. This approach is dependent on the concept of the *commutator matrix* associated to the nilpotent $\mathbb{Z}$-Lie algebra. As a result, they were able to compute the faithful dimension for nilpotency classes $c = 2$ and $c = 3$.

Following Bardestani et al. rank minimization method, we obtain the faithful dimension of the free nilpotent $\mathbb{Z}$-Lie algebra of class $c = 4$ on n generators. An explicit description of the commutator matrix is obtained by using the *Hall basis* of the free $\mathbb{Z}$-Lie algebra $\mathfrak{f}_{n,4}$.

We also explore the computation of the faithful dimension for nilpotency class $c = 5$. With the aid of computer-assisted symbolic computations, we obtain an upper bound for $m_{\text{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q))$ of magnitude $n^5 q^4$.