TORONTO SUMMER SCHOOL EXERCISES

JUNE HUH

Chow ring. Fix a rational simplicial fan Σ on N with primitive ray generators u_ρ.

Exercise 1. Let S_Σ be the polynomial ring with variables x_ρ indexed by the rays of Σ, and define

$$A^*(\Sigma) := S_\Sigma/(I_\Sigma + J_\Sigma),$$

where I_Σ is the ideal generated by the quadratic monomials

$$x_{\rho_1} x_{\rho_2} \cdots x_{\rho_k}, \ \ \{\rho_1, \rho_2, \ldots, \rho_k\} \text{ does not generate a cone in } \Sigma,$$

and J_Σ is the ideal generated by the linear forms

$$\sum_\rho \langle u_\rho, m \rangle \, x_\rho, \ \ m \text{ is an element of the dual space } N^\vee.$$

Show that the graded component $A^k(\Sigma)$ is spanned by degree k squarefree monomials in S_Σ.

Balancing condition.

Exercise 2. Define balancing condition for a k-dimensional weight on a rational fan. How should one define the balancing condition for weights on fans with irrational rays?

Exercise 3. Let P be a rational polytope containing 0 in its interior, and let Σ be the fan obtained by taking the cone over the edges of P.

(1) Show that Σ supports a positive 2-dimensional balanced weight.

(2) Assume hereafter that P is simple. Compute the dimensions of the vector spaces

$$\text{MW}_k(\Sigma) := \{k\text{-dimensional balanced weights on } \Sigma\}.$$

(3) Does the Chow ring $A^*(\Sigma)$ satisfies Poincaré duality?

(4) Generically perturb the direction of a ray in Σ, and again compute dim $\text{MW}_k(\Sigma)$.

Exercise 4. Let Q be the standard octahedron with vertices

$$(\pm 1, 0, 0), \ (0, \pm 1, 0), \ (0, 0, \pm 1),$$

and let Σ be the fan obtained by taking the cone over the proper faces of Q. Compute

$$\dim \text{MW}_0(\Sigma), \quad \dim \text{MW}_1(\Sigma), \quad \dim \text{MW}_2(\Sigma), \quad \dim \text{MW}_3(\Sigma).$$

Does $A^*(\Sigma)$ satisfies Poincaré duality? Answer the same when Q is the standard icosahedron.

Exercise 5. Find an isomorphism $A^k(\Sigma)^\vee \simeq \text{MW}_k(\Sigma)$ for a simplicial fan Σ.

Matroids. Let E be a finite set, and let M be a matroid on E. We introduce vector spaces

$$\mathbb{R}^E := \bigoplus_{i \in E} \mathbb{R}\mathbf{e}_i \text{ and } N_E := \mathbb{R}^E / \langle \mathbf{e}_E \rangle,$$

where $\mathbf{e}_E$ is the sum of the $\mathbf{e}_i$'s for $i \in E$.

Exercise 6. Give one definition of "matroid on E". According to that definition, what is a flat?

Suppose that the empty set is a flat of M. The Bergman fan Σ_M is the collection of cones

$$\mathrm{cone}(\mathbf{e}_{F_1}, \ldots, \mathbf{e}_{F_k}) \subseteq N_E, \text{ one for each flag of flats } F_1 \subsetneq \cdots \subsetneq F_k,$$

where $\mathbf{e}_F$ is the sum of the $\mathbf{e}_i$'s for $i \in F$.

Exercise 7. Prove that a top-dimensional weight on Σ_M is balanced if and only if it is constant.

Exercise 8. Describe $A^*(\Sigma_\mathrm{M})$ in terms of M.

Exercise 9. Let M be a uniform matroid of rank 4 on $E = \{0, 1, 2, 3, 4\}$. Compute the dimensions

$$\dim \mathrm{MW}_0(\Sigma_\mathrm{M}), \quad \dim \mathrm{MW}_1(\Sigma_\mathrm{M}), \quad \dim \mathrm{MW}_2(\Sigma_\mathrm{M}), \quad \dim \mathrm{MW}_3(\Sigma_\mathrm{M}).$$

Does $A^*(\Sigma_\mathrm{M})$ satisfies Poincaré duality?

Exercise 10. Consider any two maximal flag of nonempty proper flats of M:

$$\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r \subsetneq E \quad \text{and} \quad \varnothing \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r \subsetneq E$$

Show that the corresponding monomials agree in the Chow ring of Σ_M:

$$\prod_{i=1}^r x_{F_i} = \prod_{i=1}^r x_{G_i} \in A^*(\Sigma_\mathrm{M}).$$

Exercise 11. We say that a subposet $\mathcal{L} \subseteq 2^E$ satisfies the partition property if:

($*$) for every F in $\mathcal{L}$, each element of $E \backslash F$ is in exactly one subset that covers F in $\mathcal{L}$.

Prove that $\mathcal{L}$ is the collection of flats of a matroid iff every "interval" of $\mathcal{L}$ satisfies ($*$).

Hard Lefschetz theorem and Hodge-Riemann relations.

Exercise 12. Let Σ be a complete fan of dimension 2.

(1) Formulate and prove the Poincaré duality for $A^*(\Sigma)$.

(2) Formulate and prove the hard Lefschetz theorem for $A^*(\Sigma)$.

(3) Formulate and prove the Hodge-Riemann relations for $A^*(\Sigma)$.

Exercise 13. A real-valued function c on 2^E is said to be *strictly submodular* if $c_\varnothing = c_E = 0$ and

$$c_{I_1} + c_{I_2} > c_{I_1 \cap I_2} + c_{I_1 \cup I_2} \text{ for any two incomparable subsets } I_1, I_2 \subseteq E.$$

Construct a strictly submodular function on 2^E.

Exercise 14. Let Σ_{M} be the Bergman fan of a rank 3 matroid.

(1) Formulate and prove the Poincaré duality for $A^*(\Sigma_{\mathrm{M}})$.

(2) Formulate and prove the hard Lefschetz theorem for $A^*(\Sigma_{\mathrm{M}})$.

(3) Formulate and prove the Hodge-Riemann relations for $A^*(\Sigma_{\mathrm{M}})$.

Find a smooth projective surface X over $\mathbb{C}$ whose Chow ring satisfies $A^*(X) \simeq A^*(\Sigma_{\mathrm{M}})$.

Exercise 15. Given M, is there a smooth projective variety X over $\mathbb{C}$ satisfying $A^*(X) \simeq A^*(\Sigma_{\mathrm{M}})$?

Tropical Laplacian. Let Σ be a fan in $\mathbb{R}^n$ with fixed ray generators $\mathbf{u}_i$. A 2-dimensional weight w on Σ is said to be *geometrically balanced* if, for each ray i of Σ, there is a real number d_i satisfying

$$d_i \mathbf{u}_i = \sum_{i \sim j} w_{ij} \mathbf{u}_j,$$

where the sum is over all the neighbors j of i in Σ. The *tropical Laplacian* of such w is the square matrix L_w defined by

$$(L_w)_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -w_{ij} & \text{if } i \sim j, \\ 0 & \text{if } i \nsim j. \end{cases}$$

Exercise 16. How is the corank of L_w related to the dimension of $\mathbb{R}^n$?

Exercise 17. Let w be the unique 2-dimensional positive weight on the Bergman fan of a rank 3 matroid. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

Exercise 18. Let w be the unique 2-dimensional positive weight on the fan over the edges of the standard cube. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

Exercise 19. Let w be any 2-dimensional positive weight on the fan over the edges of the standard octahedron. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

Let Σ_D be the 2-dimensional "Desargues" fan on $N_{\{0,1,2,3,4\}}$ with ray generators

$$\mathbf{e}_{01},\ \mathbf{e}_{02},\ \mathbf{e}_{03},\ \mathbf{e}_{04},\ \mathbf{e}_{12},\ \mathbf{e}_{13},\ \mathbf{e}_{14},\ \mathbf{e}_{23},\ \mathbf{e}_{24},\ \mathbf{e}_{34},$$
$$\mathbf{e}_{234},\ \mathbf{e}_{134},\ \mathbf{e}_{124},\ \mathbf{e}_{123},\ \mathbf{e}_{034},\ \mathbf{e}_{024},\ \mathbf{e}_{023},\ \mathbf{e}_{014},\ \mathbf{e}_{013},\ \mathbf{e}_{012},$$

whose 2-dimensional cones correspond to inclusions between the subsets of $\{0, 1, 2, 3, 4\}$.

Exercise 20. Let w be the unique 2-dimensional positive weight on the above fan Σ_D. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

Exercise 21. Is there a 2-dimensional positive balanced weight w in $\mathbb{R}^3$ whose tropical Laplacian has more than one negative eigenvalue? How about in $\mathbb{R}^4$?

Intersection theory on matroids.

Exercise 22. Give one definition of the characteristic polynomial $\chi_{\mathrm{M}}(q)$, and compute the characteristic polynomial of uniform matroids.

Exercise 23. Let M be the rank 3 uniform matroid on $E = \{0, 1, 2, 3\}$, and set

$$\alpha_{\mathrm{M}} = x_0 + x_{01} + x_{02} + x_{03}, \quad \beta_{\mathrm{M}} = x_1 + x_2 + x_3 + x_{12} + x_{13} + x_{23}.$$

Show that the following equalities hold in the Chow ring $A^*(\Sigma_{\mathrm{M}})$:

$$\alpha_{\mathrm{M}}^2 = 1 \cdot x_0 x_{01}, \quad \alpha_{\mathrm{M}}\beta_{\mathrm{M}} = 3 \cdot x_0 x_{01}, \quad \beta_{\mathrm{M}}^2 = 3 \cdot x_0 x_{01}.$$

Compare the numbers with the reduced characteristic polynomial $\chi_{\mathrm{M}}(q)/(q-1)$.

Exercise 24. Let M be a rank 3 matroid. Show that, for any $\ell_1 \in A^1(\Sigma_{\mathrm{M}})$,

$$\begin{vmatrix} \ell_1\ell_1 & \ell_1\ell_2 \\ \ell_1\ell_2 & \ell_2\ell_2 \end{vmatrix} \leqslant 0,$$

if $A^*(\Sigma_{\mathrm{M}})$ satisfies Hodge-Riemann relations with respect to $\ell_2 \in A^1(\Sigma_{\mathrm{M}})$.