Problem Set #1

Due: 15 September 2011

1. There is a useful way of describing the points of the closed interval $[a,b]$. As usual, we assume that $a < b$.

(a) Consider the interval $[0,b]$, for $b > 0$. Prove that if x lies in $[0,b]$, then we have $x = tb$ for some t with $0 \leq t \leq 1$. What is the significance of the number t? What is the midpoint of the interval $[0,b]$?

(b) Prove that if $x \in [a,b]$, then we have $x = (1-t)a + tb$ for some t with $0 \leq t \leq 1$. What is the midpoint of the interval $[a,b]$? What is the point $1/3$ of the way from a to b?

(c) Prove conversely that if $0 \leq t \leq 1$ then $(1-t)a + tb$ is in $[a,b]$.

2. If X and Y are sets, then the ***union*** $X \cup Y$ is the set consisting of all elements that are in either X and Y (or in both X and Y). The ***intersection*** of X and Y is the set $X \cap Y$ consisting of all elements that are in both X and Y. The empty set, denoted by $\varnothing$, is the set that contains no element.

Describe each of the following subsets of $\mathbb{R}$ as a union of intervals.

(a) $A = \{x \in \mathbb{R} : x^2 + 4x + 13 < 0\} \cap \{x \in \mathbb{R} : 3x^2 + 5 > 0\}$

(b) $B = \{x \in \mathbb{R} : (x+2)(x-1)(x-5) < 0\} \cap \left\{x \in \mathbb{R} : \dfrac{3x+1}{x-2} \geq 0\right\}$

(c) $C = \left\{x \in \mathbb{R} : \dfrac{x^2 - 5x + 4}{x^2 - 9} < 0\right\} \cup \{x \in \mathbb{R} : \sqrt{7x+1} + x = 17\}$

3. Let x_1, x_2, y_1, and y_2 be four real numbers.

(a) Show that $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$.

(b) Prove the Cauchy-Schwarz inequality:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}. \tag{CS}$$

(c) Deduce that equality holds in (CS) only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.