# Problem Set #3

Due: 29 September 2011

**1.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is ***even*** if $f(x) = f(-x)$ and ***odd*** if $f(x) = -f(-x)$. For example, the functions $x \mapsto x^2$, $x \mapsto |x|$ and $x \mapsto \cos(x)$ are even, while $x \mapsto x$ and $x \mapsto \sin(x)$ are odd.

**(a)** In the four cases obtain by choosing $f\colon \mathbb{R} \to \mathbb{R}$ to be even or odd, and $g\colon \mathbb{R} \to \mathbb{R}$ to be even or odd, determine whether $f+g$ is even, odd, or not necessarily either.

**(b)** Do the same for $fg$.

**(c)** Do the same for $f \circ g$.

**2.** Using the definition of a limit, establish the following:

**(a)** Show that $\lim_{x\to 1} \sqrt{x} = 1$.

**(b)** For any $a \in \mathbb{R}$, show that $\lim_{x\to a} \sin(x) = \sin(a)$.

**Hint.** The following sum-product trigonometric identity may be useful:

$$\sin(x) - \sin(a) = 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right).$$

**3.** **(a)** Use the definition of the limit to prove that $\lim_{x\to 0} \frac{1}{x} \neq \ell$ for every real number $\ell$.

**(b)** Suppose that $\lim_{x\to a} f(x) = \ell$. If $\ell > 0$, then show that there exists a neighbourhood $(a-r, a+r)$ of $a$ such that $f$ is positive on $(a-r, a+r) \setminus \{a\}$.