Problem Set #7

Due: Thursday, 27 October 2011

1. Construct counterexamples for the following statements.

(a) If a function $g(x)$ is differentiable at $x = a$ and a function $f(x)$ is not differentiable at $g(a)$, then the function $(f \circ g)(x)$ is not differentiable at $x = a$.

(b) If a function $g(x)$ is not differentiable at $x = a$ and a function $f(x)$ is differentiable at $g(a)$, then the function $(f \circ g)(x)$ is not differentiable at $x = a$.

(c) If a function $g(x)$ is not differentiable at $x = a$ and a function $f(x)$ is not differentiable at $g(a)$, then the function $(f \circ g)(x)$ is not differentiable at $x = a$.

2. If the function f is three times differentiable and $D[f] \neq 0$ then the ***Schwarzian derivative*** of f at x, denoted $S[f]$, is defined to be

$$S[f] := \frac{D^3[f]}{D[f]} - \frac{3}{2}\left(\frac{D^2[f]}{D[f]}\right)^2 .$$

(a) Let $f(x) := \dfrac{ax+b}{cx+d}$ where a, b, c, and d are constants satisfying $ad - bc \neq 0$. Show that $S[f] = 0$.

(b) For functions g and h, show that $S[g \circ h] = \big(S[g] \circ h\big)\big(D[h]\big)^2 + S[h]$.

3. Let P be the population of a certain region as a function of time t. The rate of change of this population depends on the current population and is given by

$$\frac{dP}{dt} = kP(\ell - P)\,,$$

for positive constants k and ℓ.

(a) For what nonnegative values of P is the populations increasing? Decreasing? For what values of P does the population remain constant?

(b) Find $\dfrac{d^2P}{dt^2}$ as a function of P.