Problems 09

Due: Friday, 17 March 2023 before 17:00 EST

P9.1. **(i)** Prove that $\mathbb{Z}/\langle 60\rangle$ is isomorphic to $\mathbb{Z}/\langle 3\rangle \times \mathbb{Z}/\langle 4\rangle \times \mathbb{Z}/\langle 5\rangle$.

(ii) Exhibit elements e_1, e_2, and e_3 in $\mathbb{Z}/\langle 60\rangle$ such that

$$e_1^2 = e_1 \qquad e_2^2 = e_2 \qquad e_3^2 = e_3 \qquad e_2\, e_3 = 0 \qquad e_1\, e_3 = 0 \qquad e_1\, e_2 = 0$$

and $[1]_{60} = e_1 + e_2 + e_3$.

P9.2. Consider two multiplicative subsets D and E a commutative ring R satisfying $D \subseteq E$. Let $\varphi\colon R[D^{-1}] \to R[E^{-1}]$ be the ring homomorphism defined, for any fraction r/d in $R[D^{-1}]$, by $\varphi(r/d) = r/d$. Prove that the following statements are equivalent:

(a) The map φ is a ring isomorphism.

(b) For any element e in E, the fraction $e/1$ is a unit in $R[D^{-1}]$.

(c) For any element e in E, there exists an element s in R such that $e\,s \in D$.

P9.3. Prove that $\mathbb{Z}/\langle 512\rangle$ has exactly one maximal ideal.