Solutions 08

Each quotient ring R/I in the top row of Table 1 is isomorphic to a ring S in the bottom row.

Table 1. Table of quotient rings and rings

$\dfrac{R}{I}$	$\dfrac{\mathbb{Z}[x]}{\langle 18, 81, x\rangle}$	$\dfrac{\mathbb{Z}[x]}{\langle x^2+4x+5\rangle}$	$\dfrac{\mathbb{R}[x]}{\langle x^2-2\rangle}$	$\dfrac{\mathbb{R}[x]}{\langle x^2+2\rangle}$	$\dfrac{\mathbb{C}[x]}{\langle x^2\rangle}$	$\dfrac{\mathbb{Q}[x]}{\langle x^2-2\rangle}$						
S	$\mathbb{Z}$	$\dfrac{\mathbb{Z}}{\langle 3\rangle}$	$\dfrac{\mathbb{Z}}{\langle 9\rangle}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}\times\mathbb{R}$	$\mathbb{Z}[\mathrm{i}]$	$\mathbb{Q}[x]$	$\mathbb{Q}[\sqrt{2}]$	$\mathbb{R}[x]$	$\left\{\begin{bmatrix} z & w \\ 0 & z\end{bmatrix} \,\middle|\, z, w \in \mathbb{C}\right\}$

P8.1. Match each of the quotient rings $\dfrac{\mathbb{Z}[x]}{\langle 18, 81, x\rangle}$ and $\dfrac{\mathbb{Z}[x]}{\langle x^2+4x+5\rangle}$ with a ring S by exhibiting an explicit ring isomorphism.

Solution. We claim that
$$\frac{\mathbb{Z}[x]}{\langle 18, 81, x\rangle} \cong \frac{\mathbb{Z}}{\langle 9\rangle}\,.$$
The evaluation map $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}$, defined for any f in $\mathbb{Z}[x]$ by $\varphi(f) = \mathrm{ev}_{x=0}(f) = f(0)$, and the canonical map $\pi\colon \mathbb{Z} \to \mathbb{Z}/\langle 9\rangle$, defined for any integer m by $\pi(m) = [m]_9$, are ring homomorphisms. Hence, the composite map $\pi\,\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}/\langle 9\rangle$ is also a ring homomorphism. For any integer m, the polynomial $m \in \mathbb{Z}[x]$ having degree 0 maps to $[m]_9$, so the composite map $\pi\,\varphi$ is surjective. Since $\mathrm{Ker}(\pi) = \langle 9\rangle$ and $\mathrm{Ker}(\varphi) = \langle x\rangle$, it follows that $\mathrm{Ker}(\pi\,\varphi) = \langle 9, x\rangle$. The First Isomorphism Theorem establishes that $\widetilde{\pi\,\varphi}\colon \mathbb{Z}[x]/\langle 9, x\rangle \to \mathbb{Z}/\langle 9\rangle$ is a ring isomorphism.

It remains to show that $\langle 18, 81\rangle = \langle 9\rangle$ in $\mathbb{Z}$. Since 9 divides both 18 and 81, we have $\langle 18, 81\rangle \subseteq \langle 9\rangle$. Conversely, the equation $(5)(18) + (-1)(81) = 9$ implies that $\langle 18, 81\rangle \supseteq \langle 9\rangle$. Since $\langle 18, 81\rangle = \langle 9\rangle$, we also have $\mathbb{Z}[x]/\langle 9, x\rangle = \mathbb{Z}[x]/\langle 18, 81, x\rangle$.

Since $x^2 + 4x + 5 = (x + 2 + \mathrm{i})(x + 2 - \mathrm{i})$ in $\mathbb{C}[x]$, we next claim that
$$\frac{\mathbb{Z}[x]}{\langle x^2+4x+5\rangle} \cong \mathbb{Z}[\mathrm{i}]\,.$$
The evaluation map $\theta\colon \mathbb{Z}[x] \to \mathbb{Z}[\mathrm{i}]$, defined for any polynomial f in $\mathbb{Z}[x]$ by
$$\theta(f) := \mathrm{ev}_{x=-2-\mathrm{i}}(f) = f(-2-\mathrm{i})\,,$$
is a ring homomorphism. For any integers a and b, the polynomial $(a + 2b) - b\,x$ in $\mathbb{Z}[x]$ maps to $(a - 2b) - b(-2 - \mathrm{i}) = a + b\,\mathrm{i}$, so the map θ is surjective. Since the polynomial $x^2 + 4x + 5$ has $-2 + \mathrm{i}$ as a root, we see that $\langle x^2 + 4x + 5\rangle \subseteq \mathrm{Ker}(\theta)$. Conversely, for any polynomial $f \in \mathrm{Ker}(\theta)$, division with remainder shows that there is a polynomial q in $\mathbb{Z}[x]$ and an element and r in $\mathbb{Z}$ such that $f = (x^2 + 4x + 5)q + r$. As $\theta(f) = 0 = \theta(x^2 + 4x + 5)$ and $\theta(r) = r$, we see that $r = 0$, $f \in \langle x^2 + 4x + 5\rangle$, and $\mathrm{Ker}(\theta) \subseteq \langle x^2 + 4x + 5\rangle$. Therefore, the First Isomorphism Theorem establishes that $\overline{\theta}\colon \mathbb{Z}[x]/\langle x^2 + 4x + 5\rangle \to \mathbb{Z}[\mathrm{i}]$ is a ring isomorphism.

Remark. The evaluation map $\psi\colon \mathbb{Z}[x] \to \mathbb{Z}[\mathrm{i}]$ defined, for any rational polynomial f, by $\psi(f) := f(-2 + i)$ also induces the desired isomorphism.

$\square$

P8.2. Match each of the quotient rings $\dfrac{\mathbb{R}[x]}{\langle x^2-2\rangle}$ and $\dfrac{\mathbb{R}[x]}{\langle x^2+2\rangle}$ with a ring S by exhibiting an explicit ring isomorphism.

Solution. Since $x^2-2 = (x-\sqrt{2})(x+\sqrt{2})$ in $\mathbb{R}[x]$, we claim that

$$\frac{\mathbb{R}[x]}{\langle x^2-2\rangle} \cong \mathbb{R}\times\mathbb{R}\,.$$

The map $\varphi\colon \mathbb{R}[x]\to\mathbb{R}\times\mathbb{R}$ defined, for any polynomial f in $\mathbb{R}[x]$, by

$$\varphi(f) = \Big(\mathrm{ev}_{x=\sqrt{2}}(f), \mathrm{ev}_{x=-\sqrt{2}}(f)\Big) = \Big(f(\sqrt{2}), f(-\sqrt{2})\Big),$$

is ring homomorphism, because it is just a product of evaluation maps. For any real numbers a and b, we have

$$\varphi\left(\frac{\sqrt{2}a}{4}(x+\sqrt{2})\right) = \left(\frac{\sqrt{2}a}{4}(\sqrt{2}+\sqrt{2}), -\frac{\sqrt{2}a}{4}(-\sqrt{2}+\sqrt{2})\right) = (a,0)$$
$$\varphi\left(-\frac{\sqrt{2}b}{4}(x-\sqrt{2})\right) = \left(-\frac{\sqrt{2}b}{4}(\sqrt{2}-\sqrt{2}), -\frac{\sqrt{2}b}{4}(-\sqrt{2}-\sqrt{2})\right) = (0,b)\,,$$

so the map φ is surjective. Moreover, a real polynomial has $\pm\sqrt{2}$ as a root if and only if it is divisible by the polynomial $x\mp\sqrt{2}$. Hence, we deduce that

$$\mathrm{Ker}(\varphi) = \big\langle x-\sqrt{2}\big\rangle \cap \big\langle x+\sqrt{2}\big\rangle = \langle x^2-2\rangle\,.$$

Therefore, the First Isomorphism Theorem establishes that $\overline{\varphi}\colon \mathbb{Z}[x]/\langle x^2-2\rangle \to \mathbb{R}\times\mathbb{R}$ is a ring isomorphism.

Since $x^2+2 = (x-\sqrt{2}\,\mathrm{i})(x+\sqrt{2}\,\mathrm{i})$ in $\mathbb{C}[x]$, we next claim that

$$\frac{\mathbb{R}[x]}{\langle x^2+2\rangle} \cong \mathbb{C}\,.$$

The evaluation map $\psi\colon \mathbb{R}[x]\to\mathbb{C}$, defined for any polynomial f by

$$\psi(f) = \mathrm{ev}_{x=\sqrt{2}\,\mathrm{i}}(f) = f\big(\sqrt{2}\,\mathrm{i}\big),$$

is a ring homomorphism. For any real numbers a and b, the image of the polynomial $a+(\sqrt{2})^{-1}bx$ is $a+b\,\mathrm{i}$, so the map φ is surjective. Since the polynomial x^2+2 has $\sqrt{2}\,\mathrm{i}$ as a root, we see that $\langle x^2+2\rangle \subseteq \mathrm{Ker}(\varphi)$. Conversely, for any polynomial $f\in\mathrm{Ker}(\varphi)$, division with remainder shows that there exists polynomials q and r in $\mathbb{R}[x]$ such that $f=(x^2+2)q+r$ and $\deg(r)<2$ or $r=0$. As $\varphi(f)=0=\varphi(x^2+2)$, we see that $\varphi(r)=0$. Since $\sqrt{2}\,\mathrm{i}$ is not a real number, there is no nonzero polynomial of degree less than 2 having it as a root. Hence, we deduce that $r=0$, $f\in\langle x^2+2\rangle$, and $\mathrm{Ker}(\varphi)\subseteq\langle x^2+2\rangle$. Therefore, the First Isomorphism Theorem proves that $\widetilde{\varphi}\colon \mathbb{R}[x]/\langle x^2+2\rangle\to\mathbb{C}$ is a ring isomorphism. $\square$

Remark. The evaluation map $\eta\colon \mathbb{R}[x]\to\mathbb{C}$, defined for any real polynomial f by $\eta(f) := f\big(-\sqrt{2}\,\mathrm{i}\big)$, also induces the desired isomorphism.

P8.3. Match each of the quotient rings $\dfrac{\mathbb{C}[x]}{\langle x^2 \rangle}$ and $\dfrac{\mathbb{Q}[x]}{\langle x^2 - 2 \rangle}$ with a ring S by describing an explicit ring isomorphism.

Solution. We claim that

$$\frac{\mathbb{C}[x]}{\langle x^2 \rangle} \cong \left\{ \begin{bmatrix} z & w \\ 0 & z \end{bmatrix} \;\middle|\; z, w \in \mathbb{C} \right\}.$$

Consider the evaluation map φ from $\mathbb{C}[x]$ to the ring of (2×2)-matrices defined by $\varphi(c_m\, x^m + c_{m-1}\, x^{m-1} + \cdots + c_1\, x + c_0) = c_m\, \mathbf{N}^m + c_{m-1}\, \mathbf{N}^{m-1} + \cdots + c_1\, \mathbf{N} + c_0\, \mathbf{N}^0$ where $\mathbf{N} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. For any complex numbers z and w, we have

$$\varphi(w\, x + z) = w \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & w \\ 0 & z \end{bmatrix},$$

so the map φ is surjective. As

$$\mathbf{N}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that $\mathbf{N}^m = 0$ for all integers m greater than 1. We deduce that $\langle x^2 \rangle = \operatorname{Ker}(\varphi)$. Thus, the First Isomorphism Theorem proves that

$$\overline{\varphi} \colon \mathbb{C}[x]/\langle x^2 \rangle \to \left\{ \begin{bmatrix} z & w \\ 0 & z \end{bmatrix} \;\middle|\; z, w \in \mathbb{C} \right\}$$

is a ring isomorphism.

Since $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ in $\mathbb{R}[x]$, we also claim that

$$\frac{\mathbb{Q}[x]}{\langle x^2 - 2 \rangle} \cong \mathbb{Q}[\sqrt{2}].$$

The evaluation map $\psi \colon \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$, defined for any polynomial f in $\mathbb{Q}[x]$, by

$$\psi(f) = \operatorname{ev}_{x=\sqrt{2}}(f) = f(\sqrt{2}),$$

is ring homomorphism. For any real numbers a and b, we have $\psi(a + b\, x) = a + b\sqrt{2}$, so the map φ is surjective. Since the polynomial $x^2 - 2$ has $\sqrt{2}$ as a root, we see that $\langle x^2 - 2 \rangle \subseteq \operatorname{Ker}(\varphi)$. Conversely, for any polynomial $f \in \operatorname{Ker}(\varphi)$, division with remainder shows that there are polynomials q and r in $\mathbb{Q}[x]$ such that $f = (x^2-2)q + r$ and $\deg(r) < 2$ or $r = 0$. As $\varphi(f) = 0 = \varphi(x^2 - 2)$, we see that $\varphi(r) = 0$. Since $\sqrt{2}$ is not a rational number, there is no nonzero polynomial of degree less than 2 having it as a root. Hence, we deduce that $r = 0$, $f \in \langle x^2 - 2 \rangle$, and $\operatorname{Ker}(\varphi) \subseteq \langle x^2 - 2 \rangle$. Therefore, the First Isomorphism Theorem proves that $\overline{\varphi} \colon \mathbb{Q}[x]/\langle x^2 - 2 \rangle \to \mathbb{Q}[\sqrt{2}]$ is a ring isomorphism. □

Remark. One can also replace the map ψ with the evaluation map $\mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ sending x to $-\sqrt{2}$.