Problem Set #6 Due: Friday, 27 March 2020

1. (a) Using the Robinson–Schensted–Knuth algorithm, determine the permutation that corresponds the tableaux pair

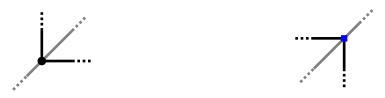
$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 8 \\ 4 & 9 \\ 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 8 \\ 3 & 5 & 9 \\ 4 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

(b) Using Viennot's construction, to find the tableaux pairs corresponding to the permutations

 $\sigma \coloneqq 2 \ 3 \ 7 \ 9 \ 1 \ 6 \ 5 \ 4 \ 8 \in \mathfrak{S}_9 \qquad \text{and} \qquad \tau \coloneqq 8 \ 4 \ 5 \ 6 \ 1 \ 9 \ 7 \ 3 \ 2 \in \mathfrak{S}_9.$

- 2. Consider a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma = \sigma^{-1}$, and let (P,Q) denote the corresponding tableaux pair. A fixed point in the permutation σ is an index *i* such that $\sigma_i = i$.
 - (a) If the standard tableau *P* has shape λ , then demonstrate that the number of columns having odd length equals the alternating sum of the parts: $\sum_{i \ge 1} (-1)^{i+1} \lambda_i$.
 - (b) For any positive integer *i* and all $1 \le j \le \lambda_i$, let $L_j^{(i)}$ denote the shadow lines in the Viennot construction for the permutation σ corresponding to the *i*-th rows in the tableau *P*. Prove that the number of shadow lines $L_j^{(i)}$ meeting the diagonal line x = y in a northeast corner is equal to the number of shadow lines $L_j^{(i+1)}$ meeting the diagonal line x = y is a southwest corner.

FIGURE 1. A southwest corner (left) and northeast corner (right)



- (c) Prove that the number of fixed points in σ is equal to the number of columns having odd length in *P*.
- **3.** Fix $n \in \mathbb{N}$.
 - (a) Prove that any permutation of more than n^2 elements has a monotonic subsequence of length greater than n.
 - (b) Find a formula for the number of permutations in \mathfrak{S}_{n^2} that have no monotonic subsequences of length greater than *n*.
- 4. (a) If λ is an integer partition having no hook of length 2, then prove that there exists $k \in \mathbb{N}$ such that $\lambda = (k, k-1, k-2, ..., 2, 1)$.
 - (b) For any integer partition, prove that there exists $k \in \mathbb{N}$ such that the number of odd hook lengths minus the number of even hook lengths equals $\binom{k+1}{2}$.
- **5.** Fix $n \in \mathbb{N}$.
 - (a) Show that the number of standard tableau of shape (n^2) is the Catalan number $\frac{1}{n+1} {2n \choose n}$.
 - (b) For all $0 \le k < n$, show that the number of standard tableau of shape $(n-k, 1^k)$ is the binomial coefficient $\binom{n-1}{k}$.