0 Cubic Equations

Galois theory originated in studying roots of polynomials. Before developing the general theory, this chapter focuses on a special case.

0.0 Cardano's Formulas

What are the solutions to a cubic equation? Consider

$$ax^3 + bx^2 + cx + d = 0 \qquad \text{for all } a, b, c, d \in \mathbb{C}, \text{ where } a \neq 0.$$

Rewrite this equation as

$$0 = ax^3 + bx^2 + cx + d = a\left(x^3 + \tfrac{b}{a}x^2 + \tfrac{c}{a}x + \tfrac{d}{a}\right).$$

Since $a \neq 0$, it follows that $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Hence, reducing to the monic case has no effect on the roots. In other words, we may assume without loss of generality that $a = 1$.

The quadratic formula states that the solutions of $ax^2 + bx + c = 0$ for all $a, b, c \in \mathbb{C}$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To eliminate the quadratic term, set $x := y - \frac{b}{3}$. The binomial theorem gives

This substitution generalizes 'completing the square' for quadratic polynomials.

$$x^2 = \left(y - \tfrac{b}{3}\right)^2 = y^2 - 2y\left(\tfrac{b}{3}\right) + \left(\tfrac{b}{3}\right)^2 = y^2 - \tfrac{2b}{3}y + \tfrac{b^2}{9},$$
$$x^3 = \left(y - \tfrac{b}{3}\right)^3 = y^3 - 3y^2\left(\tfrac{b}{3}\right) + 3y\left(\tfrac{b}{3}\right)^2 - \left(\tfrac{b}{3}\right)^3 = y^3 - by^2 + \tfrac{b^2}{3}y - \tfrac{b^3}{27}.$$

It follows that

$$\begin{aligned} 0 &= x^3 + bx^2 + cx + d \\ &= \left(y^3 - by^2 + \tfrac{b^2}{3}y + \tfrac{b^3}{27}\right) + b\left(y^2 - \tfrac{2b}{3}y + \tfrac{b^2}{9}\right) + c\left(y - \tfrac{b}{3}\right) + d \\ &= y^3 + \left(-\tfrac{b^2}{3} + c\right)y + \left(\tfrac{2b^3}{27} - \tfrac{bc}{3} + d\right). \end{aligned}$$

Let $p := -\frac{b^2}{3} + c$ and $q := \frac{2b^3}{27} - \frac{bc}{3} + d$. Given the roots of the *reduced cubic* $y^3 + py + q$, the roots of the monic cubic $x^3 + bx^2 + cx + d$ are simply translations by $-b/3$.

To solve the reduced cubic, we exploit the nonlinear substitution $y := z - \frac{p}{3z}$. The binomial theorem implies that

$$y^3 = z^3 - 3z^2\left(\tfrac{p}{3z}\right) + 3z\left(\tfrac{p}{3z}\right)^2 + \left(\tfrac{p}{3z}\right)^3 = z^3 - pz + \tfrac{p^2}{3z} - \tfrac{p^3}{27z^3},$$
$$y^3 + py + q = \left(z^3 - pz + \tfrac{p^2}{3z} - \tfrac{p^3}{27z^3}\right) + p\left(z - \tfrac{p}{3z}\right) + q = z^3 - \tfrac{p^3}{27z^3} + q.$$

Multiplying by z^3, we deduce that $y^3 + py + q = 0$ is equivalent to the *cubic resolvent* equation $z^6 + qz^3 - p^3/27 = 0$. Since this cubic resolvent can be rewritten as $(z^3)^2 + q(z^3) - p^3/27 = 0$, the quadratic formula gives

$$z^3 = \frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right) \quad \text{and} \quad z = \sqrt[3]{\frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)}.$$

Back substitution gives a root of the reduced cubic $y^3 + py + q = 0$ and the monic cubic $x^3 + bx^2 + cx + d = 0$.

However, before we can claim to have solved the cubic equation, there are several questions that need to be answered.

(existence) We essentially assumed that a solution exists. What justifies this assumption?

(multiplicity) A (generic) cubic equation should have three roots, but the cubic resolvent has degree 6. Why?

($z = 0$) The substitution $y := z - \frac{p}{3z}$ assumes $z \neq 0$. What happens when $z = 0$?

(roots) How does one describe square roots and cube roots of complex numbers?

We postpone the first three and concentrate on the last question.

Assume that $p \neq 0$ in $y^3+py+q = 0$. For any positive $n \in \mathbb{N}$, every nonzero complex number has n distinct nth roots. The cube root symbol denotes any of the three cube roots of the complex number under the radical. To understand the cube roots, recall that

The $\pm$ in the quadratic formula indicates that a nonzero complex number has two square roots.

$$\omega = \exp(2\pi \mathrm{i}/3) = \frac{-1+\mathrm{i}\sqrt{3}}{2} \in \mathbb{C}$$

One verifies that $\omega^3 = 1$ and $\omega^2 = \exp(4\pi\mathrm{i}/3) = \frac{-1-\mathrm{i}\sqrt{3}}{2}$.

is a third root of 1. The other cube roots are obtained by multiplying by ω and ω^2.

Let $\sqrt{q^2 + 4p^3/27}$ denote a fixed square root of $q^2 + 4p^3/27 \in \mathbb{C}$. With this choice of square root, let

$$z_1 := \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)}$$

denote a fixed cube root of $\frac{1}{2}\left(-q + \sqrt{q^2 + 4p^3/27}\right) \in \mathbb{C}$. The other cube roots are again obtained by multiplying by ω and ω^2. Since $p \neq 0$, it follows that $z_1 \neq 0$ and z_1 is a root of the cubic resolvent. Setting $z_2 := -p/3z_1$, we see that $y_1 := z_1 + z_2 = z_1 - p/3z_1$ is a root of the reduced cubic $y^3 + py + q$.

To understand z_2, observe that $z_1^3 z_2^3 = z_1^3(-p/3z_1)^3 = -p^3/27$ and

$$z_1^3\left[\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)\right] = \left[\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)\right]\left[\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)\right] = -\frac{p^3}{27}.$$

Since $z_1 \neq 0$, these formulas imply that $z_2^3 = \frac{1}{2}\left(-q - \sqrt{q^2 + 4p^3/27}\right)$. Hence, we deduce that

$$z_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} \quad \text{and} \quad z_2 = \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}$$

are cube roots such that their product is $-p/3$. It follows that the three roots of $y^3 + py + q = 0$ are given by

$$\begin{aligned} y_1 &= \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}, \\ y_2 &= \omega\sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \omega^2\sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}, \\ y_3 &= \omega^2\sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \omega\sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}. \end{aligned}$$

These are *Cardano's formulas* for the roots of the reduced cubic.

Gerolamo Cardano (1501–1576) is credited with publishing the first formula for solving cubic equations, attributing it to Scipione del Ferro (1465–1526) and Niccolò Fontana Tartaglia (1500–1557); see G. Cardano, *Artis magnae sive de regulis algebraicis, liber unus*, 2011 Italian and English translation by M. Tamborini, (1570).

0.1 Permutations of the Roots

How are solutions of the cubic resolvent $z^6 + qz^3 - p^3/27 = 0$ related to roots of the monic cubic $x^3 + bx^2 + cx + d$?

0.1.0 Problem. Use Cardano's formulas to solve $y^3 + 3y + 1 = 0$.

Recall that $\omega = \exp(2\pi \mathrm{i}/3) = \frac{-1+\mathrm{i}\sqrt{3}}{3} \in \mathbb{C}$.

Solution. Since the product of the real cube roots

$$\sqrt[3]{\tfrac{1}{2}\left(-1+\sqrt{5}\right)} \qquad \text{and} \qquad \sqrt[3]{\tfrac{1}{2}\left(-1-\sqrt{5}\right)}$$

is $-1 = -p/3$, Cardano's formula shows that the roots are

$$\begin{aligned} y_1 &= \sqrt[3]{\tfrac{1}{2}\left(-1+\sqrt{5}\right)} + \sqrt[3]{\tfrac{1}{2}\left(-1-\sqrt{5}\right)}, \\ y_2 &= \omega\sqrt[3]{\tfrac{1}{2}\left(-1+\sqrt{5}\right)} + \omega^2\sqrt[3]{\tfrac{1}{2}\left(-1-\sqrt{5}\right)}, \\ y_3 &= \omega^2\sqrt[3]{\tfrac{1}{2}\left(-1+\sqrt{5}\right)} + \omega\sqrt[3]{\tfrac{1}{2}\left(-1-\sqrt{5}\right)}. \end{aligned}$$ □

Since $\omega^2 = \overline{\omega}$, the roots y_2 and y_3 are complex conjugates.

0.1.1 Problem. Use Cardano's formulas to solve $y^3 - 3y = 0$.

Obviously, we have $y^3 - 3y = y(y-\sqrt{3})(y+\sqrt{3})$.

Solution. Since

$$\sqrt[3]{\frac{1}{2}\left(0 + \sqrt{0 + \frac{4(-3)^3}{27}}\right)} = \sqrt[3]{\mathrm{i}}$$

and $(-\mathrm{i})^3 = \mathrm{i}$, set $z_1 := -\mathrm{i}$, so $z_2 := -p/3z_1 = \mathrm{i}$. Thus, Cardano's formulas give the roots $y_1 = -\mathrm{i} + \mathrm{i} = 0$, $y_2 = \omega(-\mathrm{i}) + \omega^2(\mathrm{i}) = \sqrt{3}$, and $y_3 = \omega^2(-\mathrm{i}) + \omega(\mathrm{i}) = -\sqrt{3}$. □

The surprise is that Cardano's formula expresses the real roots of $y^3 - 3y$ in terms of complex numbers.

Although Cardano's formulas only apply to the reduced cubic, we obtain formulas for the roots of an arbitrary monic cubic as follows. When z_1 and z_2 are the cube roots in Cardano's formulas, the roots of $x^3 + bx^2 + cx + d$ are

$$\begin{aligned} x_1 &:= -\frac{b}{3} + z_1 + z_2\,, \\ x_2 &:= -\frac{b}{3} + \omega z_1 + \omega^2 z_2\,, \\ x_3 &:= -\frac{b}{3} + \omega^2 z_1 + \omega z_2\,, \end{aligned}$$

where z_1 and z_2 satisfy $z_1 z_2 = -p/3$. We also know that z_1 is a root of the cubic resolvent. Our short-term goal is to understand the relationship between x_1, x_2, x_3 and z_1, z_2.

Our derivation assumed $p \neq 0$, but one verifies that these formulas give the correct roots even when $p = 0$.

To begin, we express z_1, z_2 in terms of x_1, x_2, x_3. Observe that

$$x_1 + \omega^2 x_2 + \omega x_3 = -(1 + \omega^2 + \omega)\frac{b}{3} + 3z_1 + (1 + \omega + \omega^2)z_2\,.$$

As ω is a root of $x^3 - 1 = (x-1)(x^2+x+1)$, we see that $1 + \omega + \omega^2 = 0$, so $x_1 + \omega^2 x_2 + \omega x_3 = 3z_1$ or $z_1 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)$. Similarly, we have $z_2 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3)$. This shows that the solutions z_1, z_2 to the cubic resolvent equation can be expressed in terms of the roots of the monic cubic polynomial.

Better yet, the six roots of the cubic resolvent are z_1, z_2, ωz_1, ωz_2, $\omega^2 z_1$, and $\omega^2 z_2$. In terms of x_1, x_2, x_3, these roots are

$$\begin{aligned} z_1 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)\,, & z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_3 + \omega x_2)\,, \\ \omega z_1 &= \tfrac{1}{3}(x_2 + \omega^2 x_3 + \omega x_1)\,, & \omega z_2 &= \tfrac{1}{3}(x_3 + \omega^2 x_2 + \omega x_1)\,, \\ \omega^2 z_1 &= \tfrac{1}{3}(x_3 + \omega^2 x_1 + \omega x_2)\,, & \omega^2 z_2 &= \tfrac{1}{3}(x_2 + \omega^2 x_1 + \omega x_3)\,. \end{aligned}$$

The solutions to the cubic resolvent equation are simply obtained from z_1 by permuting the roots x_1, x_2, x_3.

The cubic resolvent has degree 6 because the symmetric group $\mathfrak{S}_3$ has order 6.

To get an even better understanding, set $D := q^2 + 4p^2/27$, so

$$z_1 = \sqrt[3]{\tfrac{1}{2}\left(-q + \sqrt{D}\right)} \qquad \text{and} \qquad z_2 = \sqrt[3]{\tfrac{1}{2}\left(-q - \sqrt{D}\right)}.$$

We claim that D can be expressed in terms of the roots x_1, x_2, x_3. Observe that

$$z_1^3 - z_2^3 = \tfrac{1}{2}(-q + \sqrt{D}) - \tfrac{1}{2}(-q - \sqrt{D}) = \sqrt{D}\,.$$

On the other hand, combining the equations

$$\begin{aligned} z_1 - z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3) - \tfrac{1}{3}(x_1 + \omega^2 x_3 + \omega x_2) \\ &= \tfrac{1}{3}(\omega^2 - \omega)(x_2 - x_3) = \tfrac{-\mathrm{i}}{\sqrt{3}}(x_2 - x_3)\,, \\ z_1 - \omega z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3) - \tfrac{1}{3}(x_3 + \omega^2 x_2 + \omega x_1) \\ &= \tfrac{1}{3}(1 - \omega)(x_1 - x_3) = \tfrac{\mathrm{i}\,\omega^2}{\sqrt{3}}(x_1 - x_3)\,, \\ z_1 - \omega^3 z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3) - \tfrac{1}{3}(x_2 + \omega^2 x_1 + \omega x_3) \\ &= \tfrac{1}{3}(1 - \omega^2)(x_1 - x_2) = \tfrac{-\mathrm{i}\,\omega}{\sqrt{3}}(x_1 - x_2)\,, \end{aligned}$$

with the factorization $z_1^3 - z_2^3 = (z_1 - z_2)(z_1 - \omega z_2)(z_1 - \omega^2 z_2)$ gives

$$\sqrt{D} = -\frac{\mathrm{i}}{3\sqrt{3}}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\,.$$

Squaring this formula yields

$$D := q^2 + \frac{4p^3}{27} = -\frac{1}{27}(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2\,.$$

The *discriminant* of $x^3 + bx^2 + cx + d$ is defined to be

$$\Delta := (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2\,.$$

The discriminant Δ is unchanged by permutations of the roots x_1, x_2, x_3.

In this notation, we have $q^2 + 4p^3/27 = -\Delta/27$ and

$$z_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{\frac{-\Delta}{27}}\right)} \qquad \text{and} \qquad z_2 = \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{\frac{-\Delta}{27}}\right)}.$$

Furthermore, the equations

$$\Delta := -4p^3 - 27q^2\,, \qquad p = -\frac{b^3}{3} + c\,, \qquad q = \frac{2b^3}{27} - \frac{bc}{3} + d\,,$$

imply that

$$\Delta = b^2c^2 + 18bcd - 4c^3 - 4b^3d - 27d^2\,.$$

We will see that any symmetric polynomial in x_1, x_2, x_3 can always be expressed in terms of the coefficients

$$b = -(x_1 + x_2 + x_3)\,, \quad c = x_1x_2 + x_1x_3 + x_2x_3\,, \quad d = -x_1x_2x_3\,.$$

0.2 Cubic Equations over the Real Numbers

What happens when the cubic equation has real coefficients? The discriminant of the reduced cubic $y^3 + py + q$ is

$$\Delta := (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2 = -4p^3 - 27q^2 .$$

The sign of the discriminant determines how many roots are real.

0.2.0 Theorem. *When the polynomial $y^3 + py + q \in \mathbb{R}[y]$ has distinct roots, there are two possibilities:*

(positive) *The roots of $y^3 + py + q$ are all real if and only if $\Delta > 0$.*

(negative) *The polynomial $y^3 + py + q$ has only one real roots (and the other two roots are complex conjugates) if and only if $\Delta < 0$.*

Proof. Suppose that y_1 is a root of the polynomial $y^3 + py + q$. It follows that

$$0 = \overline{0} = \overline{y_1^3 + py_1 + q} = (\overline{y_1})^3 + \overline{py_1} + \overline{q} = (\overline{y_1})^3 + p\overline{y_1} + q ,$$

which implies that the complex conjugate $\overline{y_1}$ is also a root.

In other words, the roots of a polynomial with real coefficients are either real or come in complex conjugate pairs.

When the roots y_1, y_2, y_3 are all real and distinct, the definition $\Delta := (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2$ shows that $\Delta > 0$. When the roots are not all real, one must be real root, say y_1, and the other form a complex conjugate pair, say y_2 and $\overline{y_2}$. Setting $y_2 = u + \mathrm{i}v$ where $u, v \in \mathbb{R}$, we obtain

$$\begin{aligned} \Delta &= \big(y_1 - (u + \mathrm{i}v)\big)^2 \big(y_1 - (u - \mathrm{i}v)\big)^2 \big((u + \mathrm{i}v) - (u - \mathrm{i}v)\big)^2 \\ &= \big((y_1 - u) - \mathrm{i}v\big)^2 \big((y_1 - u) + \mathrm{i}v\big)^2 (2\mathrm{i}v)^2 = -4v^2 \big((y_1 - u)^2 - v^2\big)^2 . \end{aligned}$$

It follows that $\Delta < 0$ when there is only one real root. □

To relate this theorem to Cardano's formulas, recall that

$$y_1 = z_1 + z_2 , \qquad y_2 = \omega z_1 + \omega^2 z_2 , \qquad y_3 = \omega^2 z_1 + \omega z_2 ,$$

where the cube roots

$$z_1 := \sqrt[3]{\tfrac{1}{2}\Big(-q + \sqrt{q^2 + \tfrac{4p^3}{27}}\Big)} \quad \text{and} \quad z_2 := \sqrt[3]{\tfrac{1}{2}\Big(-q - \sqrt{q^2 + \tfrac{4p^3}{27}}\Big)} .$$

are chosen so that $z_1 z_2 = -p/3$.

Suppose that $\Delta < 0$. The theorem implies that $y^3 + py + q = 0$ has precisely one real root and $\Delta = -4p^3 - 27q^2 < 0$. Hence, the square root $\sqrt{q^2 + 4p^3/27}$ is real, which means that we can take z_1 to be the unique real cube root. The equation $z_1 z_2 = -p/3$ implies that z_2 is also the real cube root. It follows that

$$y_1 = \sqrt[3]{\tfrac{1}{2}\Big(-q + \sqrt{q^2 + \tfrac{4p^3}{27}}\Big)} + \sqrt[3]{\tfrac{1}{2}\Big(-q - \sqrt{q^2 + \tfrac{4p^3}{27}}\Big)}$$

expresses the real root of reduced cubic in terms for real radicals. Since $\omega^2 = \overline{\omega}$, we have a complete understanding of how Cardano's formulas work when the discriminant is negative.

The case $\Delta > 0$ is different. Since $\Delta = -4p^3 - 27q^2 > 0$, one value of the square root $\sqrt{q^2 + 4p^3/27}$ is

$$\sqrt{q^2 + \frac{4p^3}{27}} = \sqrt{\frac{-\Delta}{27}} = \mathrm{i}\sqrt{\frac{\Delta}{27}} .$$

Using this, the cube roots are

$$z_1 := \sqrt[3]{\frac{1}{2}\left(-q + \mathrm{i}\sqrt{\frac{\Delta}{27}}\right)} \quad \text{and} \quad z_2 := \sqrt[3]{\frac{1}{2}\left(-q - \mathrm{i}\sqrt{\frac{\Delta}{27}}\right)}.$$

Both roots z_1 and z_2 are nonreal complex numbers when $\Delta > 0$. Nevertheless, the equation $z_1 z_2 = -p/3$ guarantees that $z_2 = \overline{z_1}$. Hence, Cardano's formulas become

$$y_1 = z_1 + \overline{z_1}\,, \qquad y_2 = \omega z_1 + \omega^2 \overline{z_1}\,, \qquad y_3 = \omega^2 z_1 + \omega \overline{z_1}\,.$$

The root y_1 is real because it is the sum of a complex number and its conjugate. Since $\omega^2 = \overline{\omega}$, the roots y_2 and y_3 are also real. We no longer have a canonical choice of z_1; it is just one cube root of a complex number. Curiously, we are using complex numbers to express the real roots of a real polynomial.

0.2.1 Problem (R. Bombelli 1550)**.** Find the roots of $y^3 - 15y - 4$.

See Book 2 of R. Bombelli, *L'algebra. Prima edizione integrale. Introduzione di Umberto Forti. Prefazione di E. Bortolotti.*, Biblioteca scientifica Feltrinelli. 13. Milano: Giangiacomo Feltrinelli Editore. lxiii, 671 p. (1966).

Solution. The discriminant is $\Delta = -4(-15)^3 - 27(-4)^2 = 13068 > 0$, so all three roots are real. Since

$$\sqrt[3]{\frac{1}{2}\left(4 + \mathrm{i}\sqrt{\frac{13068}{27}}\right)} = \sqrt[3]{2 + 11\mathrm{i}}\,,$$

and $(2+\mathrm{i})^3 = 8 + 3(4)\mathrm{i} - 3(2) - \mathrm{i} = 2 + 11\mathrm{i}$, choose $z_1 = 2 + \mathrm{i}$. Thus, Cardano's formulas give

$$\begin{aligned} y_1 &= (2+\mathrm{i}) + (2-\mathrm{i}) = 4\,, \\ y_2 &= \tfrac{1}{2}\big((-1+\mathrm{i}\sqrt{3})(2-\mathrm{i}) + (-1-\mathrm{i}\sqrt{3})(2+\mathrm{i})\big) = -2 + \sqrt{3}\,, \\ y_3 &= \tfrac{1}{2}\big((-1-\mathrm{i}\sqrt{3})(2-\mathrm{i}) + (-1+\mathrm{i}\sqrt{3})(2+\mathrm{i})\big) = -2 - \sqrt{3}\,. \end{aligned}$$ □

There is a 'purely real' solution provide ones use trigonometric functions rather than radicals.

0.2.2 Theorem (Viète 1615)**.** *Let $y^3 + py + q$ be a cubic polynomial with real coefficients. When the discriminant is positive, we have $p < 0$ and the roots are*

F. Viète, *Fontenaensis aequationum recognitione et emendatione tractatus duo*, Paris, France (1615).

$$\begin{aligned} y_1 &= 2\sqrt{\frac{-p}{3}}\cos(\theta)\,, \\ y_2 &= 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \tfrac{2\pi}{3}\right), \\ y_3 &= 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \tfrac{4\pi}{3}\right), \end{aligned}$$

where the real number θ is defined by

$$\theta := \frac{1}{3}\arccos\left(\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right).$$ ■