Problems 02

Due: Friday, 16 January 2026 before 23:59 ET

P2.1. Consider the cubic equation $x^3 + x - 2 = 0$.

i. Use Cardano's formulas (carefully) to derive the surprising formula

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}\,.$$

ii. Show that

$$1 + \frac{2}{3}\sqrt{\frac{7}{3}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{7}{3}}\right)^3,$$

and use this to explain part **i**.

P2.2. **i.** Show that $\sqrt[3]{4 + \mathrm{i}\sqrt{11}} \in \mathbb{C}$ is not of the form $a + \mathrm{i}b\sqrt{11}$ for some $a, b \in \mathbb{Z}$.

ii. Find a cubic polynomial of the form $y^3 + py + q$ with $p, q \in \mathbb{Z}$ which has

$$\sqrt[3]{4 + \mathrm{i}\sqrt{11}} + \sqrt[3]{4 - \mathrm{i}\sqrt{11}}$$

as a root.

P2.3. Consider the reduced cubic polynomial $y^3 + py + q$ with real coefficients. Assume that its discriminant is positive: $\Delta := -(4p^3 + 27q^2) > 0$.

i. Explain why $p < 0$.

ii. For a positive real number λ, the substitution $y = \lambda t$ transforms the reduced cubic equation into $\lambda^3 t^3 + \lambda p t + q = 0$, which can be expressed as

$$4t^3 - \left(\frac{-4p}{\lambda^2}\right)t - \left(\frac{-4q}{\lambda^3}\right) = 0\,.$$

Show that this coincides with $4t^3 - 3t - \cos(3\theta) = 0$ if and only if

$$\lambda = 2\sqrt{\frac{-p}{3}} \qquad \text{and} \qquad \cos(3\theta) = \frac{3\sqrt{3}q}{2p\sqrt{-p}}\,.$$

iii. Prove that

$$\left|\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right| < 1\,.$$

iv. Explain how part **iii** implies that the last equation in part **ii** can be solved for θ.

v. Show that $4t^3 - 3t - \cos(3\theta)$ has roots $\cos(\theta)$, $\cos(\theta + \frac{2\pi}{3})$, and $\cos(\theta + \frac{4\pi}{3})$.

vi. Show that the roots of $y^3 + py + q$ are

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos(\theta)\,, \qquad y_2 = 2\sqrt{\frac{-p}{3}}\cos(\theta + \tfrac{2\pi}{3})\,, \qquad y_3 = 2\sqrt{\frac{-p}{3}}\cos(\theta + \tfrac{4\pi}{3})\,.$$