# Problems 03

Due: Friday, 23 January 2026 before 23:59 ET

**P3.1.** Let $K$ denote a field of characteristic 2.

- **i.** For all $b \in K$, assume that there exists a field $L$ containing $K$ and an element $\beta \in L$ such that $b = \beta^2$. Prove that $\beta$ is the unique root of $x^2 + b$. As a consequence, we simply denote $\beta$ by $\sqrt{b}$.
- **ii.** Suppose that $f := x^2 + ax + b$ is an irreducible quadratic polynomial in $K[x]$ with $a \neq 0$. Assume that there exists a field $L$ containing $K$ and an element $\alpha \in L$ such that $\alpha$ is a root of $f$. Prove that $\alpha$ cannot be expressed in the form $u + v\sqrt{w}$ where $u, v, w \in K$.
- **iii.** For all $b \in K$, let $R(b)$ denote a root of $x^2 + x + b$ (possibly lying in some larger field). Prove that the roots of $x^2 + x + b$ are $R(b)$ and $R(b) + 1$ and explain why adding 1 to the second root gives the first.
- **iv.** Prove that the roots of $f = x^2 + ax + b$ with $a \neq 0$ are $aR(b/a^2)$ and $aR(b/a^2) + a$.

**P3.2.** Let the roots of $x^3 + 2x^2 - 3x + 5$ be $\alpha, \beta, \gamma \in \mathbb{C}$. Find univariate polynomials with integer coefficients that have the following roots.

- **i.** $\alpha\beta$, $\alpha\gamma$, and $\beta\gamma$.
- **ii.** $\alpha + 1$, $\beta + 1$, and $\gamma + 1$.
- **iii.** $\alpha^2$, $\beta^2$, and $\gamma^2$.

**P3.3.** The discriminant in $\mathbb{Z}[x_1, x_2, x_3]$ is $\Delta = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$. By analyzing the exponent vectors in the monomial expansion of $\Delta$, one may assume that there exists $c_1, c_2, \dots, c_5 \in \mathbb{Z}$ such that

$$\Delta = c_1 e_1^2 e_2^2 + c_2 e_2^3 + c_3 e_1^3 e_3 + c_4 e_1 e_2 e_3 + c_5 e_3^2 \,.$$

- **i.** The polynomial $f_1 := x^3 - 1$ has roots $1, \omega, \omega^2$. Show that $\Delta(f_1) = -27$ and deduce that $c_5 = -27$.
- **ii.** Use the polynomial $f_2 := x^3 - x$ to show that $c_2 = -4$.
- **iii.** Use the polynomial $f_3 := x^3 - 2x^2 + x$ to show that $c_1 = 1$.
- **iv.** Use the polynomial $f_4 := x^3 - 2x^2 - x + 2$ to show that $c_4 - 4c_3 = 34$.
- **v.** Use the polynomial $f_5 := x^3 - 3x^2 + 3x - 1$ to show that $c_4 + 3c_3 = 6$ and conclude that $c_4 = 18$ and $c_3 = -4$.


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